Two Online Models

Model 1. For all $t \in [T]$, both $f_t$ and $h_t$ are chosen adversarially.
- Mannor et al. (2009) provided a counterexample showing that obtaining sub-linear regret (against a benchmark action satisfying the budget constraint) and $C_T$ bounds simultaneously is impossible.
- Alternative regret metric: The $(1 - \frac{1}{T})$-regret is defined as:

$$R_T = \left(1 - \frac{1}{T}\right) \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*).$$

where:

$$x^*_t = \text{argmax}_{x \in X} \sum_{i=1}^{t} f_i(x),$$

and $X = \{x \in X : \sum_{i=1}^{t} h_i(x) \leq WB, 1 \leq t \leq T - W + 1\}$.

In other words, the benchmark action is required to satisfy the budget constraint proportionally over any window of length $W$.
- For the special case that $h_t(x)$ is linear, Sadeghi and Fazel (2020) proposed the OSPHG algorithm which obtains $O(\sqrt{W})$ and $O(\sqrt{W^3T})$ bounds for $R_T$ and $C_T$ respectively.

Algorithm 1 obtains $O(\sqrt{T})$ bounds for both $R_T$ and $C_T$ for general convex constraint functions $h_t$.

Model 2. For all $t \in [T]$, $f_t$ is chosen adversarially and $h_t$ is a random i.i.d. sample from an unknown distribution.
- Regret metric: The $(1 - \frac{1}{T})$-regret is defined as:

$$R_T = \left(1 - \frac{1}{T}\right) \sum_{t=1}^{T} f_t(x^*_t) - \sum_{t=1}^{T} f_t(x_t),$$

where:

$$x^*_t = \text{argmax}_{x \in X} \sum_{i=1}^{t} f_i(x).$$

- For the special case that $h_t(x)$ is linear, Raut et al. (2020) proposed the OFLW algorithm which obtains $O(\sqrt{T})$ bounds for $C_T$ both in expectation and with high probability, and $O(T^{3/4})$ and $O(\sqrt{T})$ regret bounds in expectation and with high probability, respectively.

Algorithm 1 obtains $O(\sqrt{T})$ bounds for both $R_T$ and $C_T$, in expectation and with high probability, for general convex constraint functions $h_t$.

Numerical Experiments

Experiment 1: Online joke recommendation using the Jester dataset.

$$f_1(x) = r^2_i x + \sum_{j=1}^{T} \theta_j^{ij} x_j x_i \forall i \in [T],$$

$$h_1(x) = (p_i^j, x) \forall i \in [T],$$

where $0 \leq r_i \leq 10$ is the rating of user $i$ for joke $j$ in the dataset.

Experiment 2: Online task assignment in crowdsourcing markets using a synthetic dataset.

$$f_1(x) = \sum_{i=1}^{T} [\log(1 + x_i) + \sum_{j=1}^{T} \theta_j^{ij} x_j x_i \forall i \in [T],$$

$$h_1(x) = (p_i^j, x) \forall i \in [T].$$

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References
Jester dataset: http://jesterarchive.ai/