Harvest Models

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Population Model

Let $N$ be the population count. A population model would have the form

$$\frac{dN}{dt} = F(N, t).$$

The simplest model is given by

$$F(N, t) = rN,$$

where $r$ is the net growth rate per unit of population. The solution $N(t) = N(0)e^{rt}$ is clearly unrealistic. Sooner or later it taxes the support system (such as food, space) and creates overcrowding (which may lead to diseases). A more reasonable model would slow down growth when $N$ is large.
Logistic Growth Model

The logistic growth model is given by

\[
\frac{dN}{dt} = F(N) = rN \left( 1 - \frac{N}{K} \right),
\]

where \( K \) is called the carrying capacity.

Instead of deriving a general solution, we can study the long time behaviour of the system by looking at the phase line diagram.

Usually we look at two things:

- Equilibrium point (Equilibria): \( N \) for which \( F(N) = 0 \).
- Stability of equilibrium points.
Phase Line of Logistic Growth Equation

Logistic Growth Model

\[ \frac{dN}{dt} \]

Population N
Estimating $r$

\[
\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right).
\]

When $N/K \ll 1$,

\[
\frac{dN}{dt} \approx rN.
\]

So we can estimate $r$ by

\[
r \approx \frac{1}{N} \frac{dN}{dt}.
\]
Let $H$ be the harvesting rate. Based on the logistic growth population model,

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - H.$$  

A model for $H$:

$$H = qEN,$$

where $E$ is the fishing effort, $q$ is the “catchability coefficient”. The new equation, now incorporating logistic growth and harvesting, is

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - qEN.$$
\( E = 0 \)
Increase $E$
More
Even More
Bifurcation
Extinction
Equilibrium population

For the harvest model,

\[
\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - qEN,
\]

setting \(\frac{dN}{dt} = 0\), we get

\[
N_1^* = 0, \quad N_2^* = K(1 - qE/r).
\]

- When \(qE < r\), \(N_2^*\) is stable.
- When \(qE \geq r\), the only stable point is \(N^* = 0\).
Sustained Yield

Under sustainable harvest ($qE < r$), the sustained yield is given by

$$H(N_2^*, E) = qE N_2^* = qEK(1 - qE/r).$$

The optimal fishing effort $E^*$ that gives the largest sustained yield is

$$E^* = \frac{r}{2q}.$$

The maximum yield is then

$$H^* = \frac{rK}{4}.$$
Economic Considerations

$p$: price per unit fish  
$c$: cost per unit fishing effort  
Then the total cost is $cE$ and the revenue is $pH$. The break even point occurs when

$$cE = pH$$
Equilibrium Fishing Effort

\[ qE^* K \left(1 - \frac{qE^*}{r}\right) = \left(\frac{c}{p}\right)E^* \]

We have

\[ E^* = \frac{r}{q} \left(1 - \frac{c}{p}\right) = \frac{r}{q} \left(1 - \frac{c}{qK}\right). \]

The ideal cost price ratio \((c/p)\) is \(qK/2\), where the maximum yield is achieved.

What happens when \(c/p \geq qK\)?

How does the model explain extinction caused by overfishing?
Depensation

Depensation is the effect on a population whereby, due to certain causes, a decrease in the breeding population leads to reduced production or survival of eggs or offspring.

To accommodate depensation effect, we introduce the following model

\[
\frac{dN}{dt} = F(N) = rN \left( \frac{N}{N_c} - 1 \right) \left( 1 - \frac{N}{K} \right),
\]

where \( N_c \) is called the “critical mass”.
Adding Harvest

\[ \frac{dN}{dt} = F(N) = rN \left( \frac{N}{N_c} - 1 \right) \left( 1 - \frac{N}{K} \right) - qEN, \]
Bifurcation

\[ E_{\text{max}} = \frac{r}{q} \left( \frac{(K + N_c)^2}{4N_cK} - 1 \right) \]
Sustained Yield Curve