Continuous Optimization Modeling

Yuan Gao

Applied Mathematics
University of Washington

yuangao@uw.edu

Winter 2015
General Form

General form of a continuous optimization problem:

\[
\begin{align*}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \leq b_i, \ i = 1, \ldots, m,
\end{align*}
\]

where \( x \in \mathbb{R}^n, f_i : \mathbb{R}^n \to \mathbb{R}, i = 0, \ldots, m. \)

This is equivalent to the maximization problem

\[
\begin{align*}
- \text{maximize} \quad & -f_0(x) \\
\text{subject to} \quad & f_i(x) \leq b_i, \ i = 1, \ldots, m.
\end{align*}
\]

Examples:

- linear program: \( f_i, i = 0, \ldots, m \) linear
- quadratic program: \( f_0 \) quadratic, \( f_i, i = 1, \ldots, m \) linear
- quadratic constrained quadratic program: \( f_i, i = 0, \ldots, m \) quadratic

Example:

\[
\begin{align*}
\text{minimize} & \quad x + y \\
\text{subject to} & \quad x \geq 0, y \geq 0.
\end{align*}
\]

Solving in CVX:

```matlab
cvx_begin
  variable x(2)
  minimize x(1) + x(2)
  subject to
    x >= 0;

  cvx_end
```
Arbitrage in derivatives market

**Stock**

Two stocks $S_1$ and $S_2$, with prices $p_1(t)$, $p_2(t)$.
Suppose you long $x_1$ shares of $S_1$ and short $x_2$ shares of $S_2$ at $t = 0$, and you close those positions at time $T$.
The initial cost is $x_1 p_1(0) - x_2 p_2(0)$.
The final value is $x_1 p_1(T) - x_2 p_2(T)$.
You’ve made a profit if $x_1 p_1(T) - x_2 p_2(T) > x_1 p_1(0) - x_2 p_2(0)$.

**Options**

A call option for an underlying asset $S_1$, with strike price $K_1$ at time $T$.
A put option for an underlying asset $S_2$, with strike price $K_2$ at time $T$.
Price of the call option and the put option are $p_c(t)$, $p_p(t)$.
Suppose you long $x_1$ shares of the call option and short $x_2$ shares of the put option at $t = 0$.
The initial cost is $x_1 p_c(0) - x_2 p_p(0)$.
The final value is $x_1 (p_1(T) - K_1)_+ - x_2 (K_2 - p_2(T))_+$. 
Arbitrage in derivatives market

For an underlying asset, we currently have $m$ call option contracts and $n$ put option contracts that expire at the same time $T$. Each option $i$ has a strike price $K_i$ and is priced at $p_i$. There is a arbitrage opportunity if there exists $x_1, \ldots, x_n$ such that

$$\sum_{i=1}^{m+n} x_i p_i < 0,$$

and

$$\sum_{i=1}^{m} x_i (S - K_i)_+ + \sum_{i=m+1}^{m+n} x_i (K_i - S)_+ \geq 0, \quad \forall S \geq 0,$$

where $S$ is the price of the underlying asset at time $T$. 
Denote $f(x, S) = \sum_{i=1}^{m} x_i(S - K_i)_+ + \sum_{i=m+1}^{m+n} x_i(K_i - S)_+$. Then the problem is equivalent to the following linear program:

\[
\text{minimize} \quad \sum_{i=1}^{m+n} x_i p_i \\
\text{subject to} \quad f(x, 0) \geq 0. \\
\quad f(x, K_i) \geq 0, \quad 1 \leq i \leq m + n, \\
\quad f(x, \max(K_i) + 1) \geq f(x, \max(K_i)).
\]

If the optimal value is smaller than zero, then there exists an arbitrage opportunity!
AMATH 383 covers many theoretical topics such as stability of dynamical systems, control theory, etc. It also covers application of these theories such as stability in the predator-prey model, controlling a car, etc. The instructor must decide how to split each lecture between theory and applications.

Denote $T_i$, $A_i$ as the fraction of the $i$th lecture devoted to theory and applications.

A certain amount of theory has to be covered before the applications can be taught. We model this as

$$A_1 + A_2 + \cdots + A_i \leq a(T_1 + T_2 + \cdots + T_i), \quad 1 \leq i \leq N,$$

where $N$ is the total number of lectures.
Course topic planning

The theory-applications split affects the emotional state of students.
Denote $s_i$ as the emotional state of a student after lecture $i$.

- $s_i = 0$, neutral
- $s_i > 0$, happy
- $s_i < 0$, unhappy

We can model the emotion dynamics of the student as

$$s_i = (1 - \theta)s_{i-1} + \theta(\alpha T_i + \beta A_i), \quad 1 \leq i \leq N,$$

where $\theta \in [0, 1]$ captures the emotional volatility of the student. $s_0$ is the initial emotional state of the student and $s_N$ is the terminal emotional state of the student.
The instructor tries to solve the following linear program:

maximize \( T_A \) \( s_N \)

subject to

\( T_i \geq 0, A_i \geq 0, T_i + A_i = 1, \quad 1 \leq i \leq N, \)
\( A_1 + A_2 + \cdots + A_i \leq a(T_1 + T_2 + \cdots + T_i), \quad 1 \leq i \leq N, \)
\( s_i = (1 - \theta)s_{i-1} + \theta(\alpha T_i + \beta A_i), \quad 1 \leq i \leq N, \)
\( s_0 = 0. \)
Course topic planning

Suppose there are $M$ students, each with a different $\alpha$ and $\beta$, then one of the things the instructor can do is

$$\text{maximize} \quad \min_{1 \leq j \leq M} s_{jN}$$

subject to

$$T_i \geq 0, A_i \geq 0, T_i + A_i = 1, \quad 1 \leq i \leq N,$$

$$s_{ji} = (1 - \theta)s_{j,i-1} + \theta(\alpha_j T_i + \beta_j A_i), \quad 1 \leq i \leq N, 1 \leq j \leq M$$

$$s_{j0} = 0, \quad 1 \leq j \leq M,$$

where $s_{ji}$ is the emotional state of the $j$th student after the $i$th class. Or perhaps

$$\text{maximize} \quad \sum_{j=1}^{M} s_{jN}$$
Reduce electricity cost

We would like to use a storage device (ex. a battery) to reduce the total cost of electricity consumed. Denote $p(t)$ as the price of electricity at time $t$, $u(t)$ as the usage at time $t$. Then the total cost is

$$\int_0^T p(t)u(t)dt.$$ 

Usually $p(t)$ is a periodic function with period as a day, so we take $T = 24$ hours. Divide a day into $N$ equal intervals, we can approximate the total cost in a day as

$$\sum_{i=1}^{N} p_i u_i.$$
Reduce electricity cost

Denote $q_i$ as the amount of electricity stored in the device at time interval $i$, and $c_i$ as the charging of the device at time interval $i$. Then we have

$$q_{i+1} = q_i + c_i,$$

where $c_i < 0$ means discharge.

Periodic boundary condition:

$$q_1 = q_N + c_N$$

For the storage device we finish with the same amount of electricity we start with.

Net consumption in period $i$ is now $u_i + c_i$. And we require it is positive (we do not pump power back into the grid).

Total cost is now

$$\sum_{i=1}^{N} p_i(u_i + c_i).$$
Reduce electricity cost

Minimizing cost can be modeled as the following linear program:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{N} p_i (u_i + c_i) \\
\text{subject to} & \quad u_i + c_i \geq 0, \quad 1 \leq i \leq N, \\
& \quad q_{i+1} = q_i + c_i, \quad 1 \leq i \leq N - 1, \\
& \quad q_1 = q_N + c_N, \\
& \quad -D \leq c_i \leq C, \quad 1 \leq i \leq N, \\
& \quad 0 \leq q_i \leq Q, \quad 1 \leq i \leq N,
\end{align*}
\]

where \( D \) is the maximum discharge rate, \( C \) is the maximum charge rate and \( Q \) is the capacity of the storage device.
Control of linear dynamical systems

Consider the system

\[ x_{i+1} = Ax_i + Bu_i, \quad 1 \leq i \leq n. \]

Example:

\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= cu.
\end{align*}
\]

Discretize the system

\[
\begin{align*}
x(t + \Delta t) &= x(t) + v \Delta t \\
v(t + \Delta t) &= v(t) + cu(t) \Delta t
\end{align*}
\]

Denote \( x(t + \Delta t) = x_{i+1}, x(t) = x_i \).

\[
\begin{align*}
x_{i+1} &= x_i + v_i \Delta t \\
v_{i+1} &= v_i + cu_i \Delta t
\end{align*}
\]

Writing in matrix form

\[
\begin{bmatrix}
x_{i+1} \\
v_{i+1}
\end{bmatrix} =
\begin{bmatrix}
1 & \Delta t \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_i \\
v_i
\end{bmatrix} +
\begin{bmatrix}
0 \\
c \Delta t
\end{bmatrix} u_i
\]
Control of linear dynamical systems

Depending on the goal, we can come up with different optimization objectives.

- Reach the goal state $x^*$ in $n$ steps. A feasibility problem.
- Reach the goal state $x^*$ in $n$ steps within minimum control cost (ex. fuel).
  \[
  \min_{x, u} \sum_{i=1}^{n} f(u_i)
  \]
- Try to reach the goal state $x^*$ in $n$ steps.
  \[
  \min_{x, u} f(x_n - x^*)
  \]
- Keep the state as close to $x^*$ as possible.
  \[
  \min_{x, u} \sum_{i=1}^{n} f(x_i - x^*)
  \]

$f$ is some penalization function, for example, $abs(e), e^2$. 
Control of linear dynamical systems

A general formulation:

\[
\text{minimize } \sum_{i=1}^{n} c(x_i, u_i)
\]

Nonlinear dynamical system is typically hard to control in this way. Not all nonlinear dynamics can be approximated well by linear dynamics. For example our favorite pendulum:

\[
\begin{align*}
\theta_{t+h} &= \theta_t + h\dot{\theta}_t \\
\dot{\theta}_{t+h} &= \dot{\theta}_t + h(-a \sin \theta_t - b\dot{\theta}_t + u)
\end{align*}
\]
Point tracking in video

Modeling image as a function

\[ \mathbb{R}^2 \rightarrow \mathbb{R} \]
Point tracking in video

Consider two consecutive images in a video $I_s$ and $I_t$. For a point $\vec{x}$, we want to find $\vec{v} \in \mathbb{R}^2$, such that

$$I_s(\vec{x} + \vec{v}) = I_t(\vec{x})$$

Assumption: A small image patch shares the same $\vec{v}$. 
Point tracking in video

The optimization problem

$$\minimize_{\vec{v} \in \mathbb{R}^2} \int_P |l_s(\vec{x} + \vec{v}) - l_t(\vec{x})|^2 d\vec{x}_1 d\vec{x}_2$$

But the function $l_s$ is nonlinear!
Linearize it:

$$f(\vec{x} + \vec{h}) \approx f(\vec{x}) + \nabla f(\vec{x})^T \vec{h}$$
Point tracking in video

\[
\min_{\vec{v} \in \mathbb{R}^2} \int_P |I_s(x + \vec{v}) - I_t(x)|^2 \, dx_1 \, dx_2 \\
\approx \min_{\vec{v} \in \mathbb{R}^2} \int_P |I_s(x) + \nabla I_s(x)^T \vec{v} - I_t(x)|^2 \, dx_1 \, dx_2 \\
= \min_{\vec{v} \in \mathbb{R}^2} \int_P |\nabla I_s(x)^T \vec{v} + I_s(x) - I_t(x)|^2 \, dx_1 \, dx_2
\]

Approximate integral by sum, setting \( dx_1 = dx_2 = 1 \):

\[
\min_{\vec{v} \in \mathbb{R}^2} \int_P |\nabla I_s(x)^T \vec{v} + I_s(x) - I_t(x)|^2 \, dx_1 \, dx_2 \\
= \min_{\vec{v} \in \mathbb{R}^2} \sum_{x \in P_h} |\nabla I_s(x)^T \vec{v} + I_s(x) - I_t(x)|^2
\]
Point tracking in video

Define

\[
A = \begin{pmatrix}
\partial_1 I_s(x_1) & \partial_2 I_s(x_1) \\
\vdots & \vdots \\
\partial_1 I_s(x_n) & \partial_2 I_s(x_n)
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
l_t(x_1) - l_s(x_1) \\
\vdots \\
l_t(x_n) - l_s(x_n)
\end{pmatrix}
\]

\[
\min_{\vec{v} \in \mathbb{R}^2} \sum_{x \in P_h} |\nabla I_s(x)^T \vec{v} + I_s(x) - l_t(x)|^2 = \min_{\vec{v} \in \mathbb{R}^2} \|A\vec{v} - b\|^2_2
\]

This becomes a linear least squares problem! The objective is not linear, but easy.
Approximating $\nabla l_s(x) = (\partial_1 l_s(x), \partial_2 l_s(x))^T$

$$\partial_1 l_s(x_1, x_2) \approx \frac{l_s(x_1 + h, x_2) - l_s(x_1, x_2)}{h}$$

$$\partial_2 l_s(x_1, x_2) \approx \frac{l_s(x_1, x_2 + h) - l_s(x_1, x_2)}{h}$$

The problem is that $(x_1, x_2)$ might not lie on the pixels!
Point tracking in video

A way of interpolating:

\[ f(y) \approx \frac{A_4}{A} f(x^1) + \frac{A_3}{A} f(x^2) + \frac{A_2}{A} f(x^3) + \frac{A_1}{A} f(x^4) \]