Deterministic Chaos

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Discrete Logistic Growth Model

For some species, population growth is a continuous process, while for others, growth takes place at discrete time intervals. This can be modelled by the following difference equation:

\[
\frac{N_{n+1} - N_n}{\Delta t} = rN_n \left( 1 - \frac{N_n}{K} \right),
\]

where \( N_n = N(n\Delta t) \).

We can also write it in terms of a map:

\[
N_{n+1} = f(N_n) = N_n + r\Delta tN_n \left( 1 - \frac{N_n}{K} \right).
\]

The general formula for \( N_n \) can be found recursively starting with an initial population \( N_0 \).
An equilibrium point $N^*$ satisfies

$$N^* = f(N^*).$$

That is,

$$N^* = N^* + r\Delta t N^* \left(1 - \frac{N^*}{K}\right).$$

Again the two equilibrium points are 0 and $K$. 
To determine the stability of an equilibrium point $N^*$, we introduce a small perturbation around $N^*$. Let

$$N_n = N^* + u_n,$$

where $|u_n| \ll 1$. Therefore we have

$$f(N_n) = f(N^* + u_n) \approx f(N^*) + f'(N^*)u_n.$$

That implies

$$u_{n+1} = f'(N^*)u_n.$$

$f'(N^*)$ is the “amplifying factor” from $u_n$ to $u_{n+1}$. For stability, its absolute value should be less than 1.
Stability of Logistic Map

Back to the discrete logistic map,

\[ N_{n+1} = f(N_n) = N_n + r\Delta t N_n \left( 1 - \frac{N_n}{K} \right). \]

The derivative

\[ f'(N) = 1 + r\Delta t - \frac{2r\Delta t N}{K}. \]

At \( N^* = 0 \), we have

\[ f'(0) = 1 + r\Delta t. \]

Stable or Unstable?

At \( N^* = K \), we have

\[ f'(K) = 1 - r\Delta t. \]

Stable or Unstable?
Cobweb Plot
\( r \Delta t = 1.9 \)
\( r \Delta t = 2.0 \)
\[ r \Delta t = 2.2 \]
$r \Delta t = 2.5$
$r\Delta t = 3.0$
$r \Delta t = 3.0, \quad n_0 = 3.0 \text{ vs. } 3.0001$