The Rayleigh-Taylor Instability

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ABSTRACT

Using linear perturbation analysis, the stability of superposed fluids is explored. Complete solutions are derived for two cases: the interface of two constant-density fluids and an exponential density profile. It is shown that a positive density gradient in the presence of a downward acceleration is unstable. Finally, these ideas are applied to the morphology of supernova remnants.

1. Introduction

The Rayleigh-Taylor instability is an important hydrodynamic effect that arises when a heavy fluid is accelerated into a light one. Similar to pouring water into oil, the heavier fluid, once perturbed, streams to the bottom, pushing the light fluid aside. This notion for a fluid in a gravitational field was first discovered by Lord Rayleigh in the 1880s and later applied to all accelerated fluids by Sir Geoffrey Taylor in 1950.

A seemingly contrived scenario, such a density structure arises frequently in astrophysics. Given a star in gravitational hydrostatic equilibrium, physical processes can produce temperature fluctuations. If the temperature at some point increases at constant pressure, the fluid density is lowered. Suddenly, we have dense material sitting on light material. The Rayleigh-Taylor instability then causes an overturn of those layers, thus resettling the structure.

Rayleigh-Taylor instabilities need not occur in a gravitational field; often a light fluid encounters a slower, heavy fluid, thereby providing a deceleration that fuels the overturn. An example of this is the interface of an expanding supernova remnant and the CSM. But before we discuss potential applications, we shall derive the general stability conditions. Most of the following derivation is taken from Chandrasekhar (1981), with additional explanations added and math filled in.

2. Formulation of the Problem

The Lagrangian equations of continuity and motion for an incompressible, inviscid fluid in the presence of gravity are, respectively,

$$\dot{\rho} + \mathbf{u} \cdot \nabla \rho = 0, \tag{1}$$

$$\rho \dot{\mathbf{u}} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \rho\, \mathbf{g}, \tag{2}$$

where p is the pressure, ρ is the density, $\mathbf{u}$ is the velocity, $\mathbf{g} = -g\hat{\mathbf{z}}$ is the gravitational acceleration directed downward, and a dot denotes a partial derivative with respect to time.

We assume for this problem that the pressure and density are stratified and that the fluids are in static equilibrium before the perturbation with (1) and (2) satisfied:

$$p = p(z), \qquad \rho = \rho(z), \qquad \mathbf{u} = 0. \tag{3}$$

We then slightly perturb the pressure and density so that $p \to p + \delta p$, $\rho \to \rho + \delta\rho$, and $\mathbf{u}$ becomes nonzero.

Next we substitute the perturbed quantities into the conservation equations, keep only the first-order perturbation terms, and subtract the equilibrium solutions. From the equation of continuity, we get

$$\frac{\partial}{\partial t}(\rho + \delta\rho) + \mathbf{u} \cdot \nabla(\rho + \delta\rho) = 0, \tag{4}$$

$$\dot{\rho} + \dot{\delta\rho} + \mathbf{u} \cdot \nabla\rho + \mathbf{u} \cdot \nabla(\delta\rho) = 0, \tag{5}$$

$$\dot{\delta\rho} + \mathbf{u} \cdot \nabla\rho = 0. \tag{6}$$

And, from the momentum equation, keeping only the first-order perturbation terms,

$$(\rho + \delta\rho)(\dot{\mathbf{u}} + \mathbf{u} \cdot \nabla\mathbf{u}) = -\nabla(p + \delta p) + (\rho + \delta\rho)\,\mathbf{g}, \tag{7}$$

$$\rho\dot{\mathbf{u}} = -\nabla(\delta p) + \delta\rho\,\mathbf{g}. \tag{8}$$

If we then take advantage of (3), then our perturbed equations reduce to

$$\dot{\delta\rho} = -w\rho', \tag{9}$$

$$\rho\dot{u} = -\frac{\partial}{\partial x}\delta p, \tag{10}$$

$$\rho\dot{v} = -\frac{\partial}{\partial y}\delta p, \tag{11}$$

$$\rho\dot{w} = -\delta p' - g\delta\rho, \tag{12}$$

where u, v, and w are the x, y, and z components of the velocity and $f' \equiv df/dz$.

In studying equilibrium, one often looks for exponential solutions, in which case solving for the exponents yields the evolution of the perturbation. For our problem, the spatial component must be bounded, so we pick

$$\mathbf{u} = \mathbf{u}_0\; e^{ik_x x + ik_y y + nt}, \tag{13a}$$

$$\delta p = \delta p_0\; e^{ik_x x + ik_y y + nt}, \tag{13b}$$

$$\delta\rho = \delta\rho_0\; e^{ik_x x + ik_y y + nt}. \tag{13c}$$

Substituting these trial solutions into (9)–(12),

$$n\delta\rho = -\rho' w, \tag{14}$$

$$n\rho u = -ik_x\delta p, \tag{15}$$

$$n\rho v = -ik_y\delta p, \tag{16}$$

$$n\rho w = -\delta p' - g\delta\rho. \tag{17}$$

Now we multiply (15) and (16) by k_x and k_y, respectively, and add them:

$$n\rho(ik_x u + ik_y v) = (k_x^2 + k_y^2)\delta p, \tag{18}$$

$$n\rho(ik_x u + ik_y v) = k^2\delta p, \tag{19}$$

where $k^2 \equiv k_x^2 + k_y^2$. But the fluid is incompressible, so

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{20}$$

or substituting our trial solution for u and v,

$$ik_x u + ik_y v = -w'. \tag{21}$$

Thus from (19),

$$k^2 \delta p = -n\rho w'. \tag{22}$$

Now combining (14) and (17) to eliminate $\delta\rho$,

$$\delta\rho = -\frac{\rho' w}{n}, \tag{23}$$

$$n\rho w = -\delta p' + \frac{g}{n}\rho' w. \tag{24}$$

We use (22) and (24) to eliminate δp:

$$\delta p = -\frac{n}{k^2}\rho w', \tag{25}$$

$$n\rho w = \left(\frac{n}{k^2}\rho w'\right)' + \frac{g}{n}\rho' w, \tag{26}$$

$$k^2 \rho w = \left(\rho w'\right)' + \frac{gk^2}{n^2}\rho' w, \tag{27}$$

$$\left(\rho w'\right)' = k^2 \rho w - \frac{gk^2}{n^2}\rho' w. \tag{28}$$

Besides the information gleaned from the conservation equations, we know that $\mathbf{u}$ is continuous at the boundary ($z = 0$), as well as w', by Eq. (21). However, other quantities may not be. Integrating (22) and (24) across an infinitesimal segment crossing the boundary gives us an additional constraint:

$$k^2 \Delta(\delta p) = -n\Delta(\rho w'), \tag{29}$$

$$\Delta(\delta p) = \frac{g}{n}\Delta(\rho w), \tag{30}$$

or

$$k^2 \frac{g}{n}\Delta(\rho w) = -n\Delta(\rho w'), \tag{31}$$

where

$$\Delta f \equiv f(0)_+ - f(0)_- \tag{32}$$

is the jump in quantities across the boundary, and the plus and minus signs denote the limits from the positive and negative sides, respectively.

Thus

$$\boxed{\left(\rho w'\right)' = k^2 w\left(\rho - \frac{g}{n^2}\rho'\right)} \tag{33}$$

describes the evolution of the fluid after the perturbation, and

$$\boxed{\Delta(\rho w') = -\frac{gk^2}{n^2}\Delta(\rho w)} \tag{34}$$

is our jump condition.

3. Example—Two Constant Density Fluids

Taking ρ_1, ρ_2 to be constant and recalling the form of our trial solutions, (33) becomes

$$w'' = k^2 w, \tag{35}$$

which can be solved with

$$w = Ae^{kz} + Be^{-kz}. \tag{36}$$

Further requiring that $w = 0$ at infinity and exploiting the continuity of w at the boundary gives

$$w_1(z) = Ae^{kz} \qquad (z < 0), \tag{37a}$$
$$w_2(z) = Ae^{-kz} \qquad (z > 0). \tag{37b}$$

Using this solution for w in (34) and noting that $w_1(0) = w_2(0)$,

$$\rho_2 w_2' - \rho_1 w_1' = -\frac{gk^2}{n^2}(\rho_2 - \rho_1) w_1(0), \tag{38}$$

$$-k\rho_2 - k\rho_1 = -\frac{gk^2}{n^2}(\rho_2 - \rho_1). \tag{39}$$

By rearranging, we get the stability criterion for two superposed, constant-density fluids:

$$\boxed{n^2 = gk\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right).} \tag{40}$$

We can see now that for $\rho_2 > \rho_1$, the trial solutions in (13a)–(13c) have real exponents and hence are unbounded with time.

4. Example—An Exponential Density Profile

Another simple case in which an analytical solution is possible is an exponential density profile. Letting

$$\rho = \rho_0 e^{az}, \tag{41}$$

(33) reduces to

$$w'' + aw' + k^2\left(\frac{ga}{n^2} - 1\right) w = 0. \tag{42}$$

This equation is solved with

$$w = Ae^{r_1 z} + Be^{r_2 z}, \tag{43}$$

where r_1 and r_2 are the roots of

$$r^2 + ar + k^2\left(\frac{ga}{n^2} - 1\right) = 0. \tag{44}$$

We now define boundary conditions. First, we confine the fluid between infinite planes at $z = 0$ and $z = h$. It follows then that w must vanish at these boundaries. From the lower boundary condition on (43),

$$B = -A, \tag{45}$$

so we have

$$w = A(e^{r_1 z} - e^{r_2 z}). \tag{46}$$

And from the upper boundary condition,

$$e^{r_1 h} - e^{r_2 h} = 0, \tag{47}$$

$$e^{(r_1 - r_2)h} = 1, \tag{48}$$

or

$$(r_1 - r_2)h = 2\pi n i. \tag{49}$$

We can now rewrite (46) in a more intuitive form by factoring and substituting (49):

$$w = Ae^{(r_1+r_2)z/2}\left[e^{(r_1-r_2)z/2} - e^{-(r_1-r_2)z/2}\right], \tag{50}$$

$$w = Ae^{(r_1+r_2)z/2}\left[e^{i\pi n z/h} - e^{-i\pi n z/h}\right], \tag{51}$$

and

$$w = Ae^{(r_1+r_2)z/2}\sin\frac{n\pi}{d}z, \tag{52}$$

or

$$w = Ae^{-az/2}\sin\frac{n\pi}{d}z, \tag{53}$$

since by (44), we have

$$r_1 + r_2 = -a. \tag{54}$$

If we employ the solutions for r_1 and r_2, we can get a condition on a:

$$r_1 = -\frac{a}{2} + \frac{1}{2}\left[a^2 + 4k^2\left(1 - \frac{ga}{n^2}\right)\right]^{1/2}, \tag{55a}$$

$$r_2 = -\frac{a}{2} - \frac{1}{2}\left[a^2 + 4k^2\left(1 - \frac{ga}{n^2}\right)\right]^{1/2}. \tag{55b}$$

Substituting these into (49) yields

$$a^2 + 4k^2\left(1 - \frac{ga}{n^2}\right) = -\frac{4n^2\pi^2}{d^2}. \tag{56}$$

Rearranging, we get our stability condition for an exponential density profile:

$$\boxed{n^2 = ag\left[1 + \frac{a^2}{4k^2} + \frac{n^2\pi^2}{k^2 d^2}\right]^{-1}.} \tag{57}$$

We find that if a is positive, the trial solution for the perturbed quantities in (13a)–(13c) are once again unbounded. In fact, the arbitrariness in our coordinate system means that a must be negative *everywhere* for the density structure to be stable.

5. Physical Application: The Interactions of Supernova Remnants with their Environments

Observations of supernova remnants (SNRs) reveal finger-shaped structures which grow as the hot, expanding ejecta collides with the circumstellar medium (CSM). Known as Rayleigh-Taylor fingers, they are thought to be caused by the Rayleigh-Taylor instability and are ubiquitous in these types of problems (see Fig. (1)).

Because of its simplicity, this particular problem is a clean example of this instability. At the SNR-CSM interface, there is no nuclear burning, degeneracy pressure, or neutrino transport; the convection timescales are long; and the reverse shock from the explosion is sufficiently far away. Given this model, we assume that the expanding SNR is the hotter, lighter gas that is decelerated in the collision with the cooler, denser CSM.

The effects of the Rayleigh-Taylor instability are very sensitive to the density profile of both the ejecta and the CSM. Previous efforts in this area have used morphological comparisons to conclude that the ejecta has a power-law density structure, rather than constant or exponential. Dwarkadas (2000) conducted a similar study on the CSM and concluded that the ambient medium follows an r^{-2} power law, indicative of a progenitor with a constant mass-loss rate. This is one example of the utility of studying Rayleigh-Taylor and other fluid instabilities.

Figs. (2) and (3) show a numerical simulation of the time evolution of the SNR-CSM interface. The Rayleigh-Taylor fingers are evident even in the early images. Time is normalized to the radius, so the only scale in the problem is the width of the interface, which is related to the scale height of the density profiles.

By comparing the last two figures, one can see the extreme sensitivity of the growth rate on the density profile of the surrounding medium. With an r^{-2} law, the instability grows much faster than with a constant-density law. With no initial disturbance, the fluids nevertheless develop the characteristic finger-like structures that become highly turbulent in a very short time. This is due mainly to viscous effects and another type of fluid instability—the Kelvin-Helmholtz instability—which tends to cause wrinkles whenever one viscous fluid layer slips past another. Consequently, observations of these and other characteristics can illuminate the underlying structure of the remnant.

6. Conclusion

We have seen that a high density fluid accelerated into a lighter one is inherently unstable and that these instabilities are manifested as finger-like extensions which precipitate mixing of the fluids. The analytical results found for two superposed, constant-density fluids and for an exponential density profile support the notion that negative density gradients are unstable. Furthermore, the Rayleigh-Taylor instability can be helpful in analyzing astrophysical flows, as was shown in the case of supernova remnants interacting with the circumstellar medium.

7. References

Chandrasekhar, S. 1981, *Hydrodynamic and Hydromagnetic Stability*, Dover Pubns., Chicago

Dwarkadas, V. V. 2000, ApJ, **541**, 418

Rayleigh, Lord 1883, Proc. Lond. Math. Soc., **14**, 170

Taylor, G. I. 1950, Proc. Roy. Soc. A, **201**, 192

"Rayleigh-Taylor Instability Experiments," http://w3.arizona.edu/~ fluidlab/project/rayliegh.html

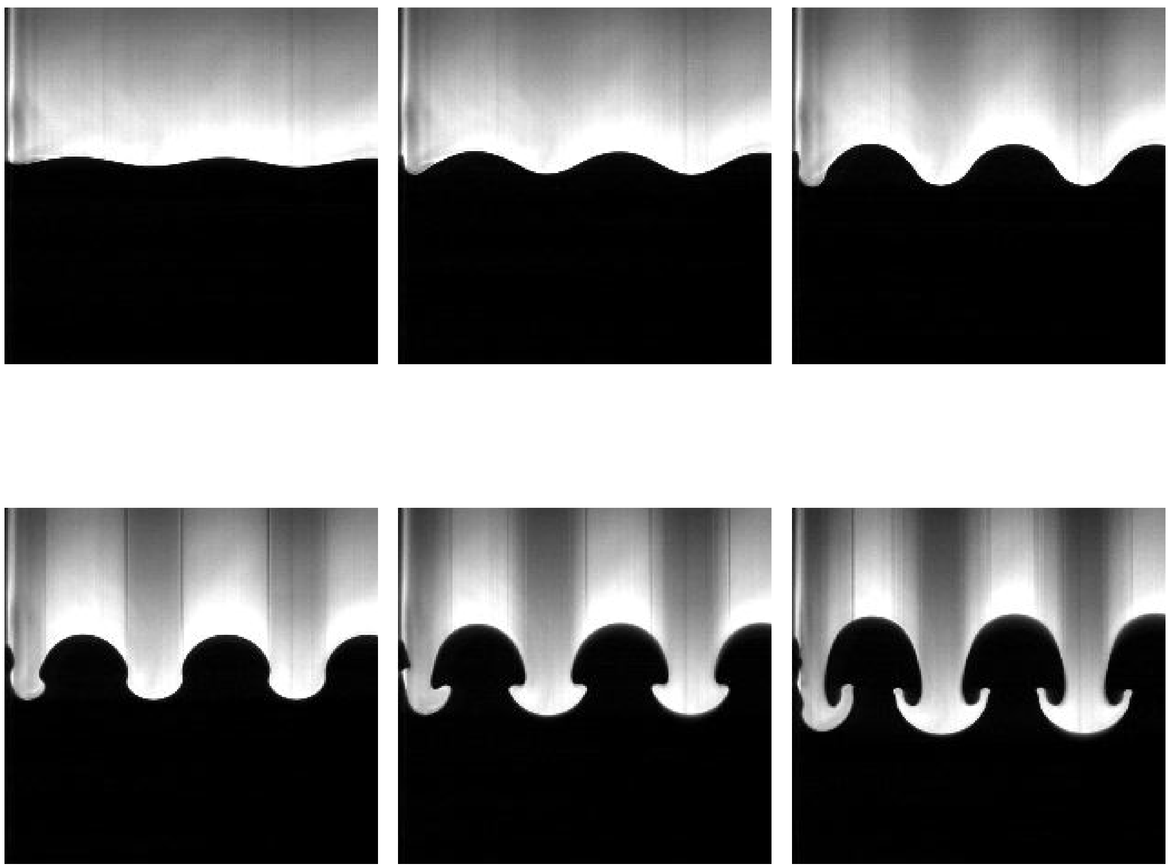

Fig. 1.— The behavior of the fluid interface given an initial sinusoidal perturbation is experimentally shown to be unstable under constant downward acceleration of $1g$. Images are from the University of Arizona (see references).

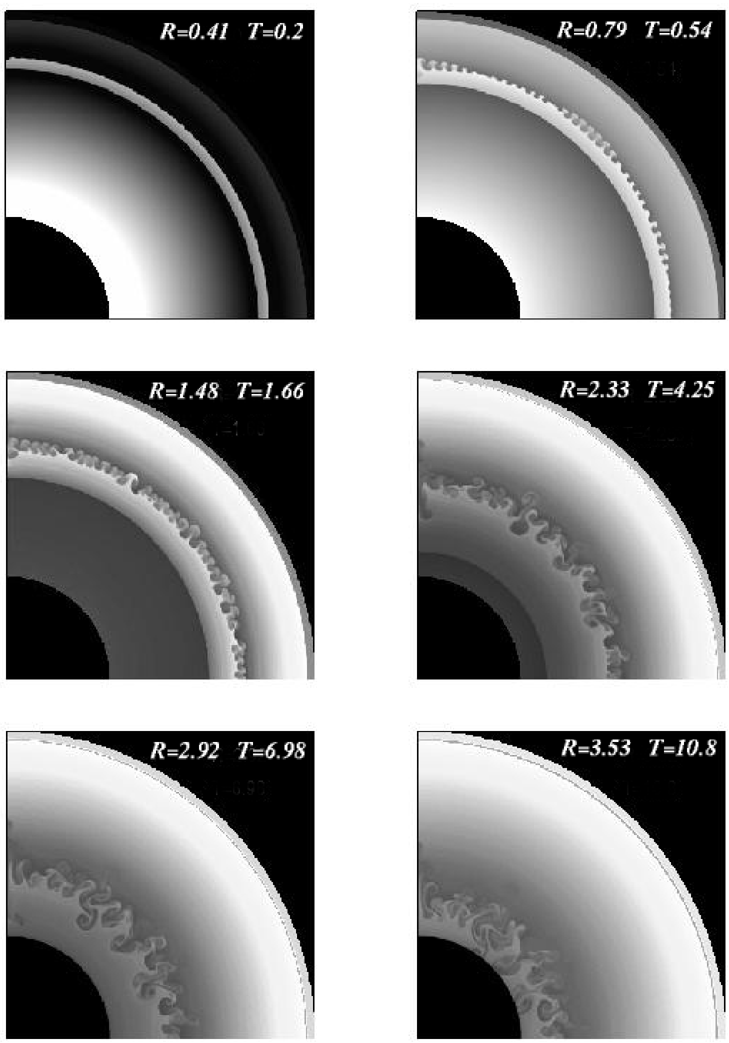

Fig. 2.— This series of images, taken from Dwarkadas (2000), shows the growth of Rayleigh-Taylor fingers at the interface of exponential-density ejecta with a constant-density medium.

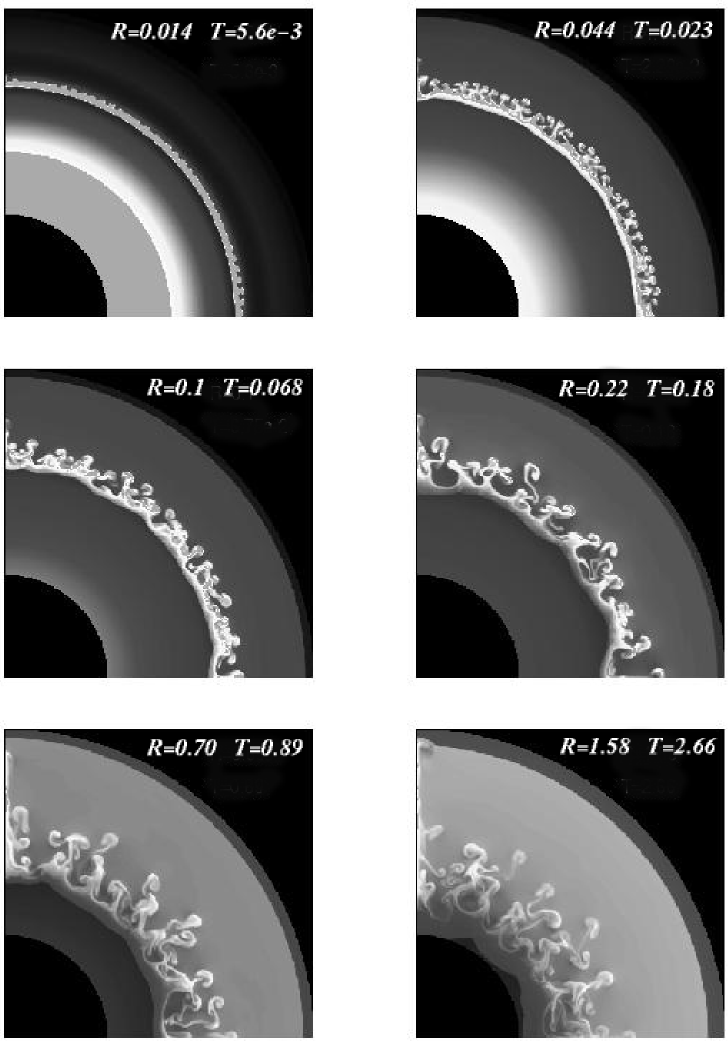

Fig. 3.— In these images, also from Dwarkadas (2000), we can see that the instability grows much more rapidly when a power-law density is used for the surrounding medium.