NOTES ON QUANTUM MECHANICS AND QUANTUM FIELD THEORY

JOSH KANTOR

1. Quantum Mechanics: Operator Formulation

As we shall see quantum mechanics has two complimentary formulations. The first one we shall look at, we call the operator formulation. First suppose that we are looking at a classical particle. If we are using the Hamiltonian description of mechanics, we can represent the path of the particle by its trajectory in phase space $(q(t), p(t))$, and we can view this as a path in $T^*(\mathbb{R}^n)$ (here $p(t)$ are the components of the momentum of the particle viewed as the components of a covector and $q(t)$ are the coordinates on $\mathbb{R}^n$, the spatial coordinates of the particle.) Observable quantities will be functions on $T^*(\mathbb{R}^n)$. For example the x coordinate of the path at some time t or the y component of momentum at time t. When we quantize we represent the state of the particle not by a path, but by an element of a Hilbert space and the observables by hermitian operators on the Hilbert space. We will use Dirac's bra-ket notation which is standard for quantum mechanics. Suppose $\mathcal{H}$ is a Hilbert space, then $|\psi\rangle$ will denote an element of the hilbert space labeled by ψ. $\langle\phi|$ will denote the element of $\mathcal{H}^*$ dual to $|\phi\rangle$ under the inner product. Finally $\langle\phi|\psi\rangle$ denotes the inner product of $|\psi\rangle$ and $|\phi\rangle$, or equivalently the pairing of $|\psi\rangle$ and $\langle\phi|$.

Now suppose we have an observable, for concreteness suppose our observable is the x component of the momentum of the particle at time t. We claim that in quantizing we should find a hermitian operator A that represents this observable. What do we mean by that. Suppose $|\psi\rangle$ is an element of the Hilbert space representing the state of particle. Let us normalize it to have unit norm. First suppose A has a complete orthonormal basis of eigenvectors $|\lambda_i\rangle$ with eigenvalue λ_i. Note that since A is hermitian, the λ_i are real. What does this have to do with physics. We postulate that the only possible measured values for the observable are the eigenvalues of A. Furthermore, the probability of measuring λ_i is $|\langle\lambda_i|\psi\rangle|^2$.

How do we go about finding $\mathcal{H}$ and the operators associated to the observables. Suppose we have a classical system described by a Hamiltonian $H(p, q)$, say $H(p, q) = \frac{p^2}{2m} + V(q)$. We had the Poisson bracket

$$\{A, B\} = \sum_{l=1}^{n} \frac{\partial A}{\partial q_l}\frac{\partial B}{\partial p_l} - \frac{\partial A}{\partial p_l}\frac{\partial B}{\partial q_l}$$

and we found that classically any observable quanity should satisfy

$$\frac{dA}{dt} = \{A, H\}.$$

In particular $\{p_i, p_j\} = 0$, $\{q_i, q_j\} = 0$, $\{q_i, p_j\} = \delta_{ij}$, which follows from Hamilton's equations of motion $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$.

It turns out we should essentially view quantization as a representation of the poisson algebra of functions (actually usually just some subalgebra), in the space of operators. Letting $\hat{A}$, denote the operator representing the observable A, we want to have $i\hbar\widehat{\{A,B\}} = [\hat{A},\hat{B}]$ mod $\mathrm{O}(\hbar^2)$, where $\hbar$ is Plank's constant. In all the basic observables we will look at there will be no terms of order higher and so we will assume units are chosen so $\hbar = 1$. By what we did above we know we will need to have $[\hat{p_i},\hat{p_j}] = [\hat{q_i},\hat{q_j}] = 0$, and $[\hat{q_i},\hat{p_j}] = i\delta_{ij}$. Also we know how the dynamics will operate since we see we will have

$$i\frac{d\hat{A}}{dt} = [\hat{A},\hat{H}].$$

This gives us the formal representation $\hat{A}(t) = e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t}$ which can be checked by formal differentiation.

In the version of quantum mechanics we have constructed, the operators carry the dynamics, they evolve in time while the state of the vector remains fixed, this is the Heisenberg picture of quantum mechanics. It is useful to have an alternative formulation so that states evolve and operators stay fixed. To do this for any fixed state $|\psi\rangle$, let $|\psi(t)\rangle = e^{-iHt}|\psi\rangle$. Then note that

$$\langle\phi(t)|\hat{A}(0)|\psi(t)\rangle = \langle\phi|e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t}|\psi\rangle = \langle\phi|\hat{A}(t)|\psi\rangle.$$

So we may think of our operator $A = A(0)$ as fixed and let the states evolve. How do they evolve, well we just compute that

$$i\frac{d|\psi(t)\rangle}{dt} = i\frac{d}{dt}e^{-i\hat{H}t}|\psi\rangle = \hat{H}e^{-i\hat{H}t}|\psi\rangle = \hat{H}|\psi(t)\rangle.$$

Shortly we will see that this is the usual Schrödinger equation.

Consider again a hamiltonian $H(p,q) = \frac{p^2}{2m} + V(q)$ where $p = (p_1,p_2,p_3)$ are the components of momentum, $p^2 = p\cdot p$, and $q = (q_1,q_2,q_3)$ are the position cooordinates. We take our Hilbert space to be $\mathcal{H} = L^2(R^3)$. We first specify how to quantize the p_i and q_i. For a function f we want to define $\hat{q_i}(f) = f(q)\cdot q_i$, $\hat{p_j}(f) = -i\frac{d}{dq_j}f(q)$. It is easy to check that the appropriate commuatation relations hold $[q_i,q_j] = [p_i,p_j] = 0$, $[q_i,p_j] = i\delta_{ij}$. If $A(p,q)$ is a nice enough function of p and q, we can define $\hat{A}$ by substituting in the $\hat{p_i},\hat{q_i}$. For example $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) = \frac{-1}{2m}\Delta + V(q)$. And we see the equations for $|\psi(t)\rangle$ is

$$\frac{d|\psi(t)\rangle}{dt} = \left(\frac{-1}{2m}\Delta + V(q)\right)|\psi(t)\rangle,$$

which is the usual Schrödinger equation.

Let us now take a step back and note that there are some issues that make things a bit more complicated than we indicated above. First our operators are unbounded and only defined on a dense subspace of $L^2(\mathbb{R}^3)$. Consequently the spectral theory is more complicated. We saw earlier that we will need to understand the eigenvectors of our operators. First consider $\hat{p_i} = -i\frac{d}{dq_i}$. Its eigenvectors are the functions $|p\rangle = e^{ip\cdot q}$ with eigenvalue p_i, which are not in L^2. Even worse the operators $\hat{q_i}$, multiplication by the i^{th} coordinate function, have no eigenvectors unless we allow distributions. Then we find that $|q\rangle = \delta(y-q)$, (as a function of y), is an eigenvector with eigenvalue q_i. It is possible to rigorously deal with the above problems by developing more of the spectral theory of unbounded operators. However this won't be necessary for anything we are doing and instead we define our way out of the problem. First we can use the following definitions extending the bra-ket notation. Whenever q represents a spatial coordinate $\langle q|\psi\rangle = \int \delta(y-q)\psi(y)dy = \psi(q)$, and if p is a momentum coordinate then $\langle p|\psi\rangle = \int e^{-ip\cdot q}\psi(q)dq = \hat{\psi}(p)$, where $\hat{\psi}(p)$ is the fourier transform of $\psi(q)$. We interpret $|\langle q|\psi\rangle|^2 = |\psi(q)|^2$ as the probability distribution for position, i.e. the probability we measure the position to lie in some region $\Omega \subset \mathbb{R}^3$ is $\int_\Omega |\psi(q)|^2 dq$. Similarly for momentum and $\langle p|\psi\rangle$.

In the case of a finite dimensional hermitian operator on a vector space, we always can choose an orthonormal basis $|v_i\rangle$ of eigenvectors and write $|v\rangle = \sum\langle a_i|v_i\rangle|v_i\rangle$. Furthermore we can always insert a complete orthonormal basis into an inner product as follows $\langle v|w\rangle = \sum\langle v|v_i\rangle\langle v_i|w\rangle$. Since in dealing with the unbounded operators above we are following our intuition for the finite dimensional case we would like to be able to formally be able to write something similar for the delta functions and exponential eigenfunctions discussed earlier. What we would like to write is $|\psi\rangle = \int dq|q\rangle\langle q|\psi\rangle$. This is true in the following sense

$$\langle\phi|\psi\rangle = \int dq\,\psi(q)\phi^*(q) = \int dq\,\langle\phi|q\rangle\langle q|\psi\rangle.$$

Similarly we want to say $|\psi\rangle = \left(\frac{1}{2\pi}\right)^3 \int dp|p\rangle\langle p|\psi\rangle$. This is true in the sense that

$$\langle\phi|\psi\rangle = \left(\frac{1}{2\pi}\right)^3 \int dp\,\hat{\psi}(p)\hat{\phi}^*(p) = \left(\frac{1}{2\pi}\right)^3 \int dp\,\langle\phi|p\rangle\langle p|\psi\rangle.$$

Thus we can think of the delta functions or exponentials as an orthonormal basis for L^2, and we use an integral as above to expand an element of the Hilbert space in this basis or to insert a basis of delta functions or exponentials inside any inner product.

2. Path Integral Formulation of Quantum Mechancis

Again consider a system with Hamiltonian of the form $H = \frac{p^2}{2m} + V(q)$. For simplicity assume we are working with a one dimensional system. We saw above that the schrodinger state vectors were of the form $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$. Thus e^{-iHt} evolves a state forward in time. If we could get some explicit information about it, that would encode most of the information about the behavior of the quantum mechanical system. So let us see if we can find an explicit representation for it. Formally we can write the following

$$\langle\phi|e^{-iH(t'-t)}|\psi\rangle = \int dqdq'\,\langle\phi|q'\rangle\langle q'|e^{-iH(t'-t)}|q\rangle\langle q|\psi\rangle.$$

Thus we should look at $\langle q'|e^{-iH(t'-t)}|q\rangle$. In fact we should first consider $\langle q'|e^{-iH\Delta t}|q\rangle$. We compute, formally of course to terms linear in Δt,

$$\begin{aligned}
\langle q'|e^{-iH\Delta t}|q\rangle &= \langle q'|e^{-i\Delta t\frac{\hat{p}^2}{2m}}e^{-i\Delta tV(\hat{q})}(1+\mathrm{O}(\Delta t^2))|q\rangle \\
&= e^{-i\Delta tV(q)}\langle q'|e^{-i\Delta t\frac{\hat{p}^2}{2m}}|q\rangle \\
&= e^{-i\Delta tV(q)}\int\frac{dpdp'}{2\pi}\langle q'|p'\rangle\langle p'|e^{-i\Delta t\hat{p}^2/2m}|p\rangle\langle p|q\rangle \\
&= e^{-i\Delta tV(q)}\int\frac{dpdp'}{2\pi}e^{-i\Delta tp^2/2m}\delta(p-p')e^{ip'q'}e^{-ipq}
\end{aligned}$$

$$= e^{-i\Delta t V(q)} \int \frac{dp}{2\pi} e^{ip(q'-q)} e^{-i\Delta t p^2/2m}.$$

We see we have ended up with something that looks like a gaussian integral. In some sense quantum mechanics is all about doing successively more complicated Gaussian like integrals. First believe, recall, check, or lookup the following identity, for $a > 0$,

$$\int dp e^{-\frac{1}{2}ap^2+Jp} = \left(\frac{2\pi}{a}\right)^{1/2} e^{\frac{J^2}{2a}}.$$

We would like to apply this to the above but our function is purely imaginary and so the above formula can't hope to work, we interpret it as follows. If we assume Δt has some negative imaginary component then we can actually do the gaussian integral, we then analytically continue to real Δt. Formally we just substitute our expressions into the formula above. We get

$$= \left(\frac{m}{2\pi i \Delta t}\right)^{\frac{1}{2}} e^{im(q'-q)^2/2\Delta t} e^{-i\Delta t V(q)}$$

Next consider a particle traveling with constant velocity from q at t_1 to q' at t_2, and set $\Delta t = t_2 - t_1$. Then $\dot{q} = \frac{(q'-q)}{\Delta t}$. If Δt is small, then V is approximately $V(q)$ for all $t \in [t_1, t_2]$. Then we see that

$$\langle q'|e^{-iH\Delta t}|q\rangle = \left(\frac{m}{2\pi i \Delta t}\right)^{\frac{1}{2}} e^{im(q'-q)^2/2\Delta t} e^{-i\Delta t V(q)}$$

$$\approx \left(\frac{m}{2\pi i \Delta t}\right)^{1/2} \exp i \int_{t_1}^{t_2} \frac{1}{2} m\dot{q}^2 - V(q) dt = \left(\frac{m}{2\pi i \Delta t}\right)^{1/2} \exp i S_{q_1,q_2}.$$

Where S_{q_1,q_2} is the action of the constant velocity path between q_1 at time t_1 and q_2 at time t_2.

Now we get to the point of the above, we have for $t' > t_n >, \ldots, > t$,

$$\langle q'|e^{-iH(t'-t)}|q\rangle =$$

$$\lim_{n\to\infty} \int dq_1, \ldots, dq_n \langle q'|e^{-iH(t'-t_n)}|q_n\rangle\langle q_n|e^{-iH(t_n-t_{n-1})}|q_{n-1}\rangle\langle q_{n-1}|e^{-iH(t_{n-1}-t_{n-2})}|q_{n-2}\rangle,$$

$$\ldots, \langle q_1|e^{-iH(t_1-t)}|q\rangle = \lim_{n\to\infty} C \int d_{q_1}, \ldots, d_{q_n} \exp i(S_{q',q_n} + S_{q_n,q_{n-1}} + \ldots + S_{q_1,q}).$$

In the above $\lim_{n\to\infty}$ denotes some limit which we won't try to make precise over increasingly fine partition of the interval $[t, t']$. C is some factor, which depends on n and the partition and arises from the infinite product of terms $\left(\frac{m}{2\pi i \Delta t}\right)^{1/2}$ in the final expression for $\langle q'|e^{-iH\Delta t}|q\rangle$. Note that for fixed $q_1, ..., q_n$, the term $S_{q',q_n} + S_{q_n,q_{n-1}} + ... + S_{q_1,q}$ is just the action of the piecewise linear path made up of consant velocity paths from q' to q_n to q_{n-1} to . . . q. By integrating over the q_i and letting our partition of $[t, t']$ become increasing fine it appears that the above is some sort of sum or integral over all paths between q' and q.

We can thus write that $\langle q'|e^{-iH(t'-t)}|q\rangle = C \int [Dq] \exp\left(i \int_t^{t'} \frac{1}{2m}\dot{q}^2 - V(q)dt\right)$. The last expression is supposed to be thought of as integration over the space of all paths of the functional that takes a path to the exponential of i times its action. $[Dq]$ denotes some formal path measure, and it really means some limit over increasingly fine piecewise smooth paths as above. Thus we see if we could figure out a way to compute these path integrals, we could simply view quantization as arising from the integral $\int [Dq] \exp iS$, where S is the classical action. As we shall see it is possible to compute these path integrals. Unfortunately proving rigorous theorems regarding path integrals is usually very difficult, but physicists have the technology to compute them. In some sense quantum mechanics is calculus in infinite dimensions. However, our understanding of this calculus is perhaps similar to Newton's or Leibnitz's understanding of finite dimensional calculus. They knew how to use it, and morally what it should mean, but not really how to put it on fully rigorous footing, we are in the same situation.

3. Functional Derivatives

Above we discovered that some sort of integration over a space of functions plays a role in quantum mechanics, thus it shouldn't seem strange we will also need to differentiate functionals on an infinite dimensional space of functions. Luckily this operation is completely mathematically rigorous as opposed to functional integrals in general. Suppose $\mathcal{S}$ is your favorite reasonable space of functions on $\mathbb{R}^m$, and $F : S \to \mathbb{R}$ is a functional. Suppose $\phi \in \mathcal{S}$. We want to construct something which should be essentially the gradient of F at ϕ. In the familiar case of a function f on $\mathbb{R}^n$ the gradient is the vector whose components are derivatives in each direction of $\mathbb{R}^n$. If we have our function ϕ we can move to a nearby function by altering ϕ at any point, thus the set of "directions" on $\mathcal{S}$ can be identified with $\mathbb{R}^m$. So the gradient should be a vector with one component for each point of $\mathbb{R}^m$, i.e. it should be another function (actually in general a distribution). What does it mean to take the derivative of F at ϕ in the "direction" of $y \in \mathbb{R}^m$. Well $\delta(x-y)$ viewed as an impulse centered at y can be thought of as a vector on $\mathcal{S}$ pointing in the "direction" that is an alteration of ϕ at y, thus our functional derivative should formally be the following

$$\frac{\delta F}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(F(\phi(x)+\epsilon\delta(x-y)) - F(\phi(x)\right).$$

Here $\frac{\delta F}{\delta\phi(y)}$ denotes the y component of the functional derivative of F at ϕ. To use the above expression the limit in epsilon must be taken before any limit operations arising from the definition of F. For example suppose $F(\psi) = \int \psi(x)^n dx$. Then

$$\frac{\delta F}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int(\phi(x)+\epsilon\delta(x-y))^n dx - \int \phi(x)^n dx\right).$$

If we formally multiply out the first integrand cancel the leading terms and them take the epsilon limit we are left with

$$\int n\phi^{n-1}(x)\delta(x-y)dx = n\phi^{n-1}(y).$$

Thus $\frac{\delta F}{\delta\phi(y)} = n\phi^{n-1}(y)$. It is easy to check that similarly if $\psi(x)$ is fixed and $F(J(x)) = \exp\int dx J(x)\psi(x)$, then

$$\frac{\delta F}{\delta J(y)} = \phi(y)\exp\int dx J(x)\phi(x).$$

Thus the basic formulas of calculus essentially hold in this setting. Let us also note that mathematically the above definition is equivalent to defining $\frac{\delta F}{\delta\phi(y)}$ to be the integral kernel of the linearization of F at ϕ.

4. Path Integral Formulation of Quantum Field Theory

We will now study the quantum theory of a scalar field. Our field is just a function ϕ on $\mathbb{R}^n$. We saw that the Klein-Gordon action $\int d^4x - \frac{1}{2}\phi(\Box + m^2)\phi$ gave rise to the equations of motion $(\Box + m^2)\phi = 0$, where $\Box = \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}$. Based on our experience that understanding path integrals is essentially equivalent to quantization, we will try to develop some technology for understanding path integrals in the case of a scalar field theory and simply hope the reader can take on faith for the moment that this will lead to some sort of physical understanding of a quantum theory of fields. Actually, what we should first try and evaluate is the following

$$Z_{\text{free}}(J) = \int [D\phi] \exp i \int d^4x - \frac{1}{2}\phi(\Box + m^2)\phi + J(x)\phi(x).$$

This is a functional of $J(x)$. The interpretation of this is that we modify the standard klein gordon equation by adding disturbance $J(x)$ which can be thought of as a source or sink for particles. If we understand $Z_{\text{free}}(J)$ for all J then we will have a detailed understanding of how our quantum system reacts to small disturbances. In figuring out what these sorts of path integrals mean it is useful to work by analogy with a finite dimensional analog. Formally we can think of

$$Z_{\text{free}}(J) = \lim_{N\to\infty} C \int d^N x \exp -\frac{i}{2} \sum_{k,l} x_k A_{kl} x_l + iJ_k x_k$$

In the above to discretize we replace the differential operator $\Box + m^2$, by a limit over matrices A_{kl}. Also, C is some normalization constant that we will deal with later. Note that formally the A depends on N though this limit is just to help us develop the theory and we won't make it precise. This is the same thing one might do in numerically solving a PDE. One could replace $\mathbb{R}^4$ by a lattice, and replace linear operators by large matrices which encoded the derivatives as finite differences, integration then becomes a sum and $J(x)$ becomes the vector J_k. We now apply the following formula

$$\int d^N x \exp -\left(\frac{1}{2}\sum_{k,l} x_k A_{kl} x_l + J_k x_k\right) = \exp \frac{1}{2}\sum_{k,l} J_k G_{kl} J_l,$$

where $G_{kl} = A^{-1}$. Of course just like when we were trying to construct the path integral for quantum mechanics we want to use a formula for a gaussian-like integral in a situation where we have an exponential with purely imaginary exponent. To deal with this rigorously we should analytic continue from a situation where the formula makese sense to the purely imaginary case. Practically we just substitute our purely imaginary expression into the formula above and using that $(iA)^{-1} = -iA^{-1}$, we obtain

$$= \lim_{N\to\infty} C \exp \frac{1}{2}\sum_{k,l}(iJ_k)(iA)^{-1}_{kl}(iJ_l) = \lim_{N\to\infty} C \exp \frac{i}{2}\sum_{k,l} J_k G_{kl} J_l.$$

We now need to figure how to un-discretize this and write things in terms of $J(x)$ again. What should the continuous version of $G_{k,l}$. It should be some sort of inverse to $\Box + m^2$, in fact it should be the Green's function for this operator. Thus we want to have $(\Box + m^2)G(x-y) = \delta(x-y)$. Taking the fourier transform of this equation we see we need to solve $[\left[-(k_0^2-k_1^2-k_2^2-k_3^2)+m^2\right]\hat{G}=1$. We can almost just say that then G should be the inverse fourier transform of $\frac{-1}{k^2-m^2}$ where k^2 is $k_0^2-k_1^2-k_2^2-k_3^2$, the lorentz inner product. Unfortunately $\frac{-1}{k^2-m^2}$ has a pole, due to the lorentz inner product and so we have to be a bit more careful. It is easy to check that

$$G(x-y)=\lim_{\epsilon\to 0}\int \frac{d^4x}{(2\pi)^4}\frac{-e^{ik\cdot(x-y)}}{k^2-m^2+i\epsilon},$$

where the limit is taken in the sense of distributions. Finally we discover that we should define

$$Z_{\text{free}}(J)=C\exp\frac{i}{2}\int d^4xd^4yJ(x)G(x-y)J(y).$$

Now we think of $Z_{\text{free}}(J)$ as encoding how our system reacts to disturbances of the Klein-Gordon Lagrangian. However, if we are only interested in the Klein-Gordon lagrangain, then we really only want to know how our system reacts to small, in fact infinitesimal disturbances. Thus we really want to understand functional derivatives with respect to J.

Let us define the n-point function

$$G^{(n)}_{\text{free}}(x_1,\ldots,x_n)=(-i)^n\frac{\delta^n Z_{\text{free}}(J)}{\delta J(x_1),\ldots,\delta J(x_n)}\Bigg|_{J=0}.$$

These are also called correlation functions. Although we could compute with functional derivatives directly, it is psychologically at least easier for us to discretize and take ordinary derivatives and then un-discretize, similar to how we discovered the formula for the free partition function. In doing this as above we replace $G(x_1-x_2)$ by a matrix G_{kl}. Since $\Box+m^2$ is self adjoint, G_{kl} should be a symmetric N by N matrix for some large N. We replace $J(x)$ by a vector J_l and $\frac{\delta}{\delta J(x_1)}$ becomes $\frac{\partial}{\partial J_{i_1}}$. With this in mind, lets compute

$$\begin{aligned}G^{(2)}_{\text{free}}(x_1,x_2)&=-\frac{\delta}{\delta J(x_1)\delta J(x_2)}\exp\frac{i}{2}\int d^4xd^4yJ(x)G(x-y)J(y)\Bigg|_{J=0}\\&=-\lim_{N\to\infty}\frac{\partial}{\partial J_{i_i}\partial J_{i_2}}\exp\frac{i}{2}\sum_{k,l}J_kG_{kl}J_l\Bigg|_{J=0}\\&=-\lim_{N\to\infty}\frac{\partial}{\partial J_{i_1}}\sum_l iG_{i_2l}J_l\exp\frac{i}{2}\sum_{k,l}J_kG_{kl}J_l\Bigg|_{J=0}\end{aligned}$$

$$=-iG_{i_1i_2}\exp\frac{i}{2}\sum_{k,l}J_kG_{kl}J_l-\left(i\sum_l G_{i_1l}J_l\right)\left(i\sum_l G_{i_2l}J_l\right)\exp\frac{i}{2}\sum_{k,l}J_kG_{kl}J_l\Bigg|_{J=0}$$

$$=-\lim_{N\to\infty}iG_{i_1i_2}=-iG(x_1-x_2).$$

In computing the derivative we used that G_{kl} is symmetric.

The same sort of computation will yield

$$G^{(4)}(x_1,x_2,x_3,x_4)=(i^2)(G(x_1-x_2)G(x_3-x_4)+G(x_1-x_3)G(x_2-x_4)+G(x_1-x_4)G(x_2-x_3)).$$

We won't do the computation we just note that in computing

$$(-i)^2 \frac{\delta}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} Z_{\text{free}}(J)\bigg|_{J=0} = \lim_{N\to\infty} \frac{\partial}{\partial J_{i_1}\partial J_{i_2}\partial J_{i_3}\partial J_{i_4}} \exp \frac{i}{2}\sum_{k,l} J_k G_{k,l} J_l \bigg|_{J=0},$$

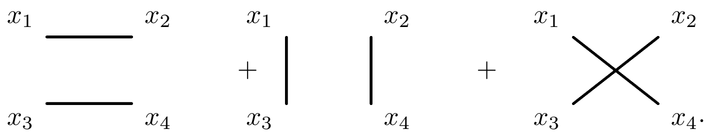
the only way to get a term that won't vanish when we set $J = 0$ is to use pairs of derivatives, one to bring down a $\sum G_{il} J_l$, and the other to get rid of the remaining J_l and sum leaving in the limit $G(x_i - x_j)$. Thus $G^{(4)}(x_1, x_2, x_3, x_4)$ will be a sum over over the pairs $G(x_i - x_j)$'s as above. Let us introduce a graphical notation for the above expression.

First we define

x_1 $x_2 = -iG(x_1 - x_2)$

Then, with the convention that terms corresponding to disconnected components of diagrams are multiplied, we can write

$$G^{(4)}(x_1, x_2, x_3, x_4) = (-i)^2 \left(G(x_1 - x_2)G(x_3 - x_4) + G(x_1 - x_3)G(x_2 - x_4) + G(x_1 - x_4)G(x_2 - x_3)\right)$$

= 

where in the last equation the diagonal lines do not intersect.

This will be more useful in computing correlation functions for the interacting theory which we do immediately. We want to compute the derivatives of

$$Z(J) = \int [D\phi] \exp i \int d^4x - \frac{1}{2}\phi(\Box + m^2)\phi - \frac{\lambda}{4!}\phi^4(x) + J(x)\phi(x).$$

Once again to deal with this we work by analogy with a discretized version. So we compute

$$Z(J) = \lim_{N\to\infty} \int d^N x \exp i \left(\sum_{k,l} x_k A_{kl} x_l - \sum_k \frac{\lambda}{4!} x_k^4 + J_k x_k \right)$$

$$= \lim_{N\to\infty} \int d^N x \exp \left(\sum_k \frac{-i\lambda}{4!} x_k^4 \right) \exp i \left(-\frac{1}{2} \sum_{k,l} x_k A_{kl} x_l + J_k x_k \right).$$

$$= \lim_{N\to\infty} \exp \left(\sum_l \frac{-i\lambda}{4!} \left(\frac{\partial}{\partial J_l} \right)^4 \right) \int d^N x \exp i \left(-\frac{1}{2} x_k A_{kl} x_l + J_k x_k \right)$$

$$= \lim_{N\to\infty} \left(1 - \frac{i\lambda}{4!} \sum_l \left(\frac{\partial}{\partial J_l} \right)^4 - \frac{\lambda^2}{4!4!2} \sum_{l,m} \left(\frac{\partial}{\partial J_l} \right)^4 \left(\frac{\partial}{\partial J_m} \right)^4 + \ldots + \right) \exp \frac{i}{2} \sum_{k,l} J_k G_{kl} J_l.$$

We have found an expression for $Z(J)$ as a series in λ. Now we will compute $G^{(2)}(x_1, x_2)$ to terms linear in λ. We will omit terms that would vanish upon setting $J = 0$ to shorten our expressions.

$$G^{(2)}(x_1,x_2) = \frac{\delta}{\delta J(x_1)\delta J(x_2)} Z(J)\Big|_{J=0}$$
$$= \lim_{N\to\infty} -\frac{\partial}{\partial J_{i_1}}\frac{\partial}{\partial J_{i_2}}\left(1+\sum_m \frac{-i\lambda}{4!}\left(\frac{\partial}{\partial J_m}\right)^4\right)\exp\frac{i}{2}\sum_{k,l} J_k G_{kl} J_l\Big|_{J=0}$$
$$= -\frac{\partial}{\partial J_{i_1}}\frac{\partial}{\partial J_{i_2}}\left(\exp\left(\frac{i}{2}\sum_{k,l} J_k G_{kl} J_l\right)+\sum_m \frac{-i\lambda}{4!}\left(-3G_{mm}^2+6iG_{mm}(iG_{ml}J_l)^2\right)\exp\left(\frac{i}{2}\sum_{k,l} J_k G_{kl} J_l\right)\right)\Big|_{J=0}$$

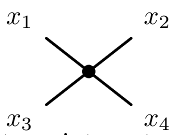
Taking the remaining derivatives is easy. The first term is just the computation that gave of the free two point function. The only way to get something nonvanishing from the G_{mm}^2 term is to bring down another $G_{i_1 i_2}$. In the last term the only thing we can do is kill the derivatives already brought down. In undiscretizing our sums are replaced by integrals and we have

$$G^{(2)}(x_1,x_2) = -iG(x_1-x_2)+\frac{\lambda}{8}\int d^4y G(0)^2 G(x_1-x_2)+\frac{\lambda}{2}\int d^4y G(y-x_1)G(y-x_2).$$

We now extend our pictorial representation to deal wih these terms. We define as before

x_1 $x_2 = -iG(x_1-x_2)$

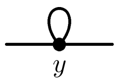
and



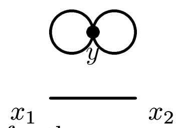
$= -i\lambda$

We further add the rule that we integrate over each internal vertex and multiply by some appropriate symmetry factor, which is basically $\frac{1}{|\mathrm{Aut}(G)|}$

For example

x_1 

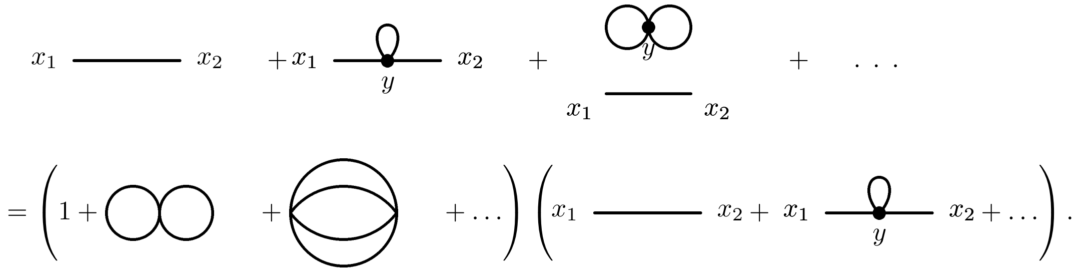
$x_2 = \frac{\lambda}{2}\int d^4y G(x_1-y)G(x_2-y)G(0)$



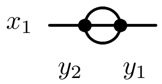
$= \frac{\lambda}{8}G(x_1-x_2)\int d^4y (G(0))^2$

Please ignore for the moment that we are encountering increasingly singular and ill defined expression, such as $G(0)$, we will explain at the end how to make sense of these.

Now we need to address some issues. First of all there were some normalizaion factors we brushed under the rug earlier, now we will come back to them. It turns out we shouled require that $Z(0) = 1$. So we should find a way to divide by our current $Z(0)$. For the two point function we have



What is the point of this. Well, we essentially want to say that $G^{(2)}(x_1, x_2)$ is a series in λ that is the sum over all Feynman diagrams with two external vertices. To enforce our requirement that $Z(0) = 1$, we omit all diagrams having a component with no external vertices. These are called vacuum bubbles. If we think of $Z(0)$, the zero point function, as Feynman diagrams with no external vertices. Then the above factorization shows that omitting vacuum bubble terms is the same as formally dividing by the original $Z(0)$. Thus our rule is that to compute $G^{(2)}(x_1, x_2)$ we sum over all terms arising from Feynman diagrams that have two external vertices and no vacuum bubbles. In general $G^{(n)}(x_1, \dots, x_n)$ is the sum over all Feynman diagrams with n external vertices, omitting vacuum terms. Here is a more complicated term



$$x_2 = \frac{-i\lambda^2}{6} \int d^4y_1 d^4y_2 G(x_1 - y_1) G^{(3)}(y_1 - y_2) G(y_2 - x_2).$$

Let us now make a couple remarks. As we saw in the case of quantum mechanics complimentary to the path integral formulation of quantization we may regard quantization as an operation which replaces classical objects by operators on a Hilbert space. In the case of a quantum field theory $\phi(x_1)$ becomes an operator valued function which we will just denote $\phi(x_1)$. What we have denoted $G^{(n)}(x_1, \dots, x_n)$ is more commonly written as $\langle 0|T[\phi(x_1), \dots, \phi(x_n)]|0\rangle$ where T denotes a time ordering that puts the $\phi(x_i)$ in increasing time order from right to left. Often this is written as $\langle \phi(x_1), \dots, \phi(x_n)\rangle$. The $|0\rangle$ denotes a vacuum state of the theory. Thus $G^{(n)}(x_1, \dots, x_n) = \langle \phi(x_1, \dots, x_n)\rangle$ is some sort of expectation value. In fact these correlation functions encode all the predictions of the theory. There are explicit formulas involving these correlation functions that can be used to predict the type of scattering phenomenon that would be observed if for example beams of particles were collided in a particle accelerator. In fact Feynman diagrams can and should be thought of literally as pictures of the processes that particles undergo. For example in the diagram above one particle is propagating then splits into three (virtual particles) which then recombine into a real particle that continues on its way. This splitting and recombing can occur anywhere so we have integrate over spacetime twice. Summing of Feynman diagrams is physically the same as summing over all possible particle events. Also it is often more convenient to work with the fourier transforms of the $G^{(n)}(x_1, \dots, x_n)$, which we will write as $\hat{G}^{(n)}(p_1, \dots, p_n)$. There are rules similar to the ones above that allow one to express $\hat{G}^{(n)}(p_1, \dots, p_n)$ as a sum over feynman diagrams. They are easy to derive from above or can be found in any book on quantum field theory.

We still haven't dealt with the fact that many of the terms we found above were quite ill defined. The procedure to deal with this is quite complicated and known as renormalization. We will give only the briefest description of what it involves. There are actually quite a few different ways of renormalizing, we describe the one that is simplest conceptually. Recall that all of the terms associated to the Feynman diagrams involve the function

$$G(x-y) = \lim_{\epsilon \to 0} \int \frac{d^4x}{(2\pi)^4} \frac{-e^{ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon}.$$

So we see that $\hat{G}^{(n)}(p_1, \ldots, p_n)$ will in particular involve terms like

$$\int d^4k \frac{-1}{k^2 - m^2 + i\epsilon},$$

where here k^2 is the lorentz innner product. We can analytically continue from real k_0 to imaginary k_0 which basically just involves replacing k^2 by $-|k|^2$ where $|k|^2$ denotes the euclidean inner product. Thus we will have $\hat{G}^{(n)}_{\text{euc}}(p_1, \ldots, p_n)$ the euclidean n point functions obtained by analytic continuation. If we can make these well defined we can analytically continue back to minkowski space to get our actual correlation functions. Now the $\hat{G}^{(n)}_{\text{euc}}(p_1, \ldots, p_n)$ are constructed out of

$$\int d^4k \frac{1}{|k|^2 + m^2}$$

where we have omitted the $i\epsilon$ because now that we are in euclidean space there are no poles. Finally we can begin the renormalization procedure. Essentially we first regularize all of our expressions by replacing

$$\int d^4k \frac{1}{|k|^2 + m^2}$$

by

$$\int_{|k|<\Omega} \frac{1}{|k|^2 + m^2}$$

which makes everything finite. Next one goes through a complicated procedure that involves analyzing how this expression diverges as $\Omega \to \infty$ and separating off a finite piece by adding counterterms that cancel out the divergences as $\Omega \to \infty$.

References

[1] Greiner, Walter. Field Quantization
[2] Rabin, J. "Introduction to Quantum Field Theory for Mathematicians", *Geometry and Quantum field theory.*