

# An Introduction to Hamiltonian Quantum Mechanics

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## Abstract

This is intended as a brief mathematician's introduction to the essential ingredients of quantum mechanics. It is essentially a write up I did to teach myself.

## 1 Classical Mechanics

The *Lagrangian* is a function describing the dynamics of the system, just like the Hamiltonian. In contrast to the Hamiltonian, which is defined as kinetic plus potential energy, the Lagrangian is kinetic minus potential. The Hamiltonian is a function of position and momentum, i.e. a function on the cotangent space  $T^*M$ . The Lagrangian is a function of position and velocity, i.e. a function on the tangent space  $TM$ .

**Example 1.1.** *The  $n$ -body problem in  $\mathbb{R}^3$*

Suppose we have  $n$  discrete particles, in configuration in  $\mathbb{R}^3$ . Then if we write the coordinates of the  $n$  particles in order,  $(x^1, y^1, z^1), \dots, (x^n, y^n, z^n)$ , we have  $3n$  coordinates (which we will label  $q^i$  for  $i = 1, \dots, 3n$ ) in  $\mathbb{R}^{3n}$ . Define  $\mathcal{C}$  to be the subset of  $\mathbb{R}^{3n}$  consisting of collisions, i.e. configurations in which two or more particles share the same coordinates. The collision set can be described as

$$\mathcal{C} = \{q \in \mathbb{R}^{3n} : (q^{3j-2}, q^{3j-1}, q^{3j}) = (q^{3k-2}, q^{3k-1}, q^{3k}), \text{ for some } j \neq k\},$$

which is the union of the  $\binom{n}{2}$  closed subsets

$$\mathcal{C}_{j,k} = \{q \in \mathbb{R}^{3n} : (q^{3j-2}, q^{3j-1}, q^{3j}) = (q^{3k-2}, q^{3k-1}, q^{3k})\}.$$

So  $\mathbb{R}^{3n} \setminus \mathcal{C}$  is an open submanifold of  $\mathbb{R}^{3n}$ . If you'd rather not think of manifolds, simply think locally: suppose the particles are configured at significant distances from one another, and only consider small movements of each particle. Then the space genuinely looks like Euclidean space (of dimension  $3n$ , not 3), and you are secretly considering the tangent space of our configuration space at a point.

Now, suppose the  $n$  particles have masses  $m_1, \dots, m_n > 0$  and forces depending only on their positions act on the particles. Denote the force on particle

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$k$  by  $(F_{3k-2}(q), F_{3k-1}(q), F_{3k}(q))$ , where  $q = (q^1, \dots, q^{3n})$ . Think of the coordinates  $q^i : \mathbb{R}^{3n} \setminus \mathcal{C} \longrightarrow \mathbb{R}$  as functions of time, so that we may examine how the system evolves over time. Newton's second law of motion,  $ma = F$ , gives, for  $i = 1, \dots, 3; k = 1, \dots, n$ , that

$$m_k \cdot (q^{3k-i})''(t) = F_{3k-i}(q).$$

As Jack notes in [1], we can compactify this notation by defining the diagonal  $3n \times 3n$  matrix  $M$  with nonzero entries  $m_1, m_1, m_1, \dots, m_n, m_n, m_n$ . Then we can write (summing the left side over  $j$ )

$$M_{ij}(q^j)''(t) = F_i(q(t)), \text{ for } i = 1, \dots, 3n.$$

Since all of the entries of  $M$  are positive, the diagonal matrix  $M$  is positive definite, i.e.  $M^* := \overline{M}^T = M$  and all of the eigenvalues of  $M$  are positive. In general, a positive definite form is a bilinear map  $B : V \times V \longrightarrow \mathbb{C}$  such that  $B(x, x) > 0$  for all nonzero  $x$ . We use  $M$  to define a positive definite form  $B_M(x, y) = M_{ij}x^iy^j$  (summing over  $i$  and  $j$ ).

We can use the positive definite form  $B_M$  to define a linear isomorphism between  $V$  and  $V^*$ . Given a vector  $v$ , we have the covector

$$B_M(v, \cdot) : V \longrightarrow \mathbb{R} : w \mapsto B_M(v, w) = M_{ij}v^iw^j.$$

Since  $M$  is positive definite, the kernel of this map is 0, and between two vector spaces of the same dimension, it must be an isomorphism. In other words, the matrix describing the masses of the particles gives us a natural way of transitioning between vectors and covectors. If we write out this correspondence in coordinates, we get  $v_i \partial / \partial x^i \mapsto M_{ij}v^j dx^i$ .

Our natural intuition is to think of velocities as vectors. If a particle has velocity  $v$ , then we are taught that the way to express momentum is as mass times velocity,  $p = mv$ . In our more sophisticated setup, we have the matrix  $M$  giving rise to a correspondence between vectors (velocities) and covectors (momenta). We can then define the momentum, by the formula

$$p_i(t) = M_{ij}(q^j)'(t) \left( = M_{ij}v^j(t) \right).$$

So we still use the masses and velocities of the system to obtain the momenta, but now we view the momentum as a covector. The cotangent bundle  $T^*N$  of a manifold  $N$  is loosely the space of pairs  $(q, p)$  where  $q$  is a point in the manifold and  $p$  is a covector based at  $q$ . Here  $N$  is our configuration space  $\mathbb{R}^{3n} \setminus \mathcal{C}$ ,  $q$  is a point in that configuration space, and  $p$  is what you get by applying the matrix  $M$  to the velocities of the particles,  $q'$ . This larger space, which consists of position and momentum, is called the phase space.

Those wishing to see the full abstract treatment should follow "The Canonical Symplectic Form on the Cotangent Bundle" in [1].

A symplectic form is a nondegenerate alternating bilinear form, usually denoted by the greek letter  $\omega : V \otimes V \longrightarrow K$  (in the context of manifolds, we also require  $d\omega = 0$ ). In other words,  $\omega(v, \cdot)$  and  $\omega(\cdot, v)$  are linear transformations of  $\cdot$ ,  $\omega(X, Y) = -\omega(Y, X)$ , and  $\omega(X, Y) = 0$  for all  $Y$  implies  $X = 0$ .

If  $\omega$  is a symplectic form on  $V$ , then there is a basis for  $V$  such that  $\omega$  is of the form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

In phase space, there is a natural symplectic form. Given  $(q, p)$ , where  $p$  is a momentum for the configuration  $q$ , we can write  $p = \xi_i dq^i$  in terms of our original coordinates, and define (summing over  $i$ )

$$\omega(X, Y) = dq^i(X)d\xi_i(Y) - d\xi_i(X)dq^i(Y).$$

Suppose we are in some generic situation where we have a symplectic form to work with, and we start with a function  $f$ . Let's just say that we're working on  $\mathbb{R}^{2n}$ , with a basis where  $\omega$  takes the form above. We can use the symplectic structure to transform that function into a vector field, kind of like the process of taking the gradient of a function. However, this transformation will give a vector that points in a direction orthogonal to the gradient, so that  $f$  is constant as we follow the vector field.

First we need to realize that the symplectic form  $\omega : V \otimes V \longrightarrow K$  can be viewed as a linear isomorphism  $\omega : V \longrightarrow V^*$ , which takes the vector  $v$  to the covector  $\omega(v, \cdot)$ . Since this is an isomorphism, we can find a vector field  $X_f$  such that  $\omega(X_f, \cdot) = df(\cdot)$ .

Usually the function  $f$  is given a proper name,  $H$ , and is called the Hamiltonian. The physical interpretation of the Hamiltonian in classical phase space is that it represents the total energy of the system. By using the process above, if we knew the total energy of a system at every point of phase space, we could produce a vector field that describes how the system must evolve over time.

In symplectic coordinates where  $\omega(x^i, y^j) = \delta^{ij}$  (and so  $\omega(y^i, x^j) = -\delta^{ij}$ ), we have any curve  $\gamma = (x^i, y^i)$  whose tangent vector is always the Hamiltonian vector field at that point must satisfy the differential equations

$$(x^j)'(t) = \frac{\partial H}{\partial y^j}(x(t), y(t)),$$

$$(y^i)'(t) = -\frac{\partial H}{\partial x^i}(x(t), y(t)).$$

These are called Hamilton's equations.

Returning to our example, we had Newton's second law of motion:

$$M_{ij}(q^j)''(t) = F_i(q(t)),$$

and our definition of momentum:

$$p_i(t) = M_{ij}(q^j)'(t).$$

Let's assume (see Jack's example for all the details) that there is a smooth function  $V$  on configuration space such that we can interpret the force as the covector  $F = -dV$ . Call  $V(q)$  the potential energy of the system, noting that it is only a function of position. Then define the kinetic energy of the system as a function of momentum,  $K(p) = 1/2 M^{ij} p_i p_j$ , where  $(M^{ij})$  is the inverse matrix of  $M = (M_{ij})$ . This is like the dual form to our  $B_M$ , something that takes covectors as arguments:  $B_M^*(p, q) = p^T M^{-1} q$ .

You can check that a curve  $q(t)$  in configuration space satisfies Newton's second law if and only if the curve  $(q(t), p(t))$  satisfies Hamilton's equations for the Hamiltonian  $H(q, p) = V(q) + K(p)$ . The law of conservation of energy is a consequence of Hamiltonian mechanics.

## 2 The Euler-Lagrange Equation

Suppose we have position coordinates  $q^i$ , that  $L(q^i, \dot{q}^i)$  is the Lagrangian of the system, and that  $L$  is at least quadratic in  $q^i, \dot{q}^i$ .

The principle of least action states that the *action*  $\int L(q^i, \dot{q}^i) dt$  must be minimized. This principle implies the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0.$$

**Example 2.1.** *The One Dimensional Euler-Lagrange Equation*

Suppose we are looking for a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$A(f) = \int_a^b F(x, f(x), f'(x)) dx$$

is minimal with boundary conditions  $f(a) = c, f(b) = d$ .

Suppose  $f_\varepsilon(x) = f(x) + \varepsilon g(x)$  is a perturbation of  $f$  by a differentiable function satisfying  $g(a) = 0 = g(b)$ . Examine the function of  $\varepsilon$

$$A(\varepsilon) = \int_a^b F(x, f_\varepsilon(x), f'_\varepsilon(x)) dx.$$

Assuming that  $F$  has continuous first partial derivatives, we know that the derivative is

$$\frac{dA}{d\varepsilon} = \int_a^b \frac{dF}{d\varepsilon}(x, f_\varepsilon(x), f'_\varepsilon(x)) dx.$$

By the chain rule,

$$\frac{dF}{d\varepsilon} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial \varepsilon} + \frac{\partial F}{\partial f_\varepsilon} \frac{\partial f_\varepsilon}{\partial \varepsilon} + \frac{\partial F}{\partial f'_\varepsilon} \frac{\partial f'_\varepsilon}{\partial \varepsilon} = 0 + \frac{\partial F}{\partial f_\varepsilon} g(x) + \frac{\partial F}{\partial f'_\varepsilon} g'(x),$$

and we can write

$$\frac{dA}{d\varepsilon} = \int_a^b \left( \frac{\partial F}{\partial f_\varepsilon} g(x) + \frac{\partial F}{\partial f'_\varepsilon} g'(x) \right) dx.$$

Since  $f$  is supposed to be an extreme value of  $A$ , when  $\varepsilon = 0$ ,  $dA/d\varepsilon = 0$  and  $f_\varepsilon = f$ . Thus we have

$$\int_a^b \left( \frac{\partial F}{\partial f} g(x) + \frac{\partial F}{\partial f'} g'(x) \right) dx = 0.$$

Using integration by parts gives

$$\int_a^b \left( \frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'} \right) g(x) dx + \left[ g(x) \frac{\partial F}{\partial f'} \right]_a^b.$$

By boundary conditions,

$$0 = \int_a^b \left( \frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'} \right) g(x) dx$$

for any differentiable function  $g$  with  $g(a) = 0 = g(b)$ . In particular, if  $g(x)$  is the product of the quantity in parentheses with  $(a-x)(x-b)$ , then the integral is non-negative, and hence the integrand must be zero, but since  $g > 0$  on  $(a, b)$ , we have the Euler-Lagrange equation

$$\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'} = 0.$$

□

The Euler-Lagrange equation implies Hamilton's equations as follows. Defining the momentum by

$$p_i = \frac{\partial L}{\partial \dot{q}^i},$$

note that

$$\dot{p}_i = \frac{\partial L}{\partial q^i}.$$

Defining the *Hamiltonian*

$$H := \sum_i \dot{q}^i p_i - L,$$

we have

$$\begin{aligned} \delta H &= \sum_i \left( p_i \delta \dot{q}^i + \delta p_i \dot{q}^i - \left( \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i + \frac{\partial L}{\partial q^i} \delta q^i \right) \right) \\ &= \sum_i \left( p_i \delta \dot{q}^i + \delta p_i \dot{q}^i - p_i \delta \dot{q}^i - \dot{p}_i \delta q^i \right) \\ &= \sum_i \left( \delta p_i \dot{q}^i - \dot{p}_i \delta q^i \right), \end{aligned}$$

and since  $\delta p_i$  and  $\delta q^i$  are independent variations, we have Hamilton's equations

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i}. \end{aligned}$$

### 3 Legendre Transformations

A *Legendre transformation* is used express functions  $f(x)$  of  $x$  as functions of the derivative of  $f$ . For example, if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a convex function

$$\frac{d^2 f}{dx^2} > 0,$$

then define the Legendre transformation  $f^*(m) = mx - f(x)$ , regarding  $x$  as a function of

$$m = \frac{df}{dx}.$$

Think of the formula  $m_0x - f^*(m_0)$  as the equation of the tangent line to  $f$  at the point  $x_0$  where  $f'(x_0) = m_0$ . To do the Legendre transform, calculate  $m = df/dx$ , solve for  $x$  and plug this in to  $mx - f^*(m)$ . Another way to look at this is to say that two functions are the Legendre transformations of each other if their derivatives are inverse functions. Note that

$$\frac{df^*}{dp}(p) = x + p \frac{dx}{dp} - \frac{df}{dx} \frac{dx}{dp} = x,$$

and

$$\frac{df}{dx}(x) = p.$$

Starting with our expression for the Hamiltonian  $H = \dot{q}p - L$  and suppressing indices, we can use Hamilton's equations to translate between  $L$  and  $H$  by a Legendre transformation. Recall that  $p = \partial L / \partial \dot{q}$ , and that  $\dot{q} = \partial H / \partial p$ , hence  $p$  is a function of  $\dot{q}$  and vice versa. Thus we have

$$H(q, p) = \dot{q} \frac{\partial L}{\partial \dot{q}} - L(q, \dot{q}) \text{ and } L(q, \dot{q}) = p \frac{\partial H}{\partial p} - H(q, p).$$

These are Legendre transformations, which change the functional dependence between velocities  $\dot{q}$  and momenta  $p$ . Formally, starting with the Lagrangian  $L : TM \rightarrow \mathbb{R}$ , define a map  $\mathbb{F}L : TM \rightarrow T^*M$  called the fiber derivative, or the Legendre transform, by

$$\mathbb{F}L(v)w = \left. \frac{d}{dt} \right|_{t=0} L(v + tw).$$

Then in finite dimensional manifolds,

$$\mathbb{F}L(q^i, \dot{q}^i) = \left( q^i, \frac{\partial L}{\partial \dot{q}^i} \right),$$

which agrees with  $p_i = \frac{\partial L}{\partial \dot{q}^i}$ . For more details, see [2].

## 4 Poisson Brackets

In general, given two functions  $f$  and  $g$  on the phase space  $T^*M$ , in terms of the variables  $q$  and  $p$  (given by the canonical symplectic form  $\omega$  on the cotangent bundle), define the Poisson bracket  $\{ \cdot, \cdot \}$  by

$$\{f, g\} = \sum_i \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right).$$

In terms of the Hamiltonian system  $(T^*M, \omega, H)$ , we could also define the Poisson bracket in terms of the symplectic form  $\omega$ , by

$$\{f, g\} = X_g f = \omega(X_f, X_g),$$

where  $X_g$  is the Hamiltonian vector field of  $g$ , defined by  $\omega(X_g, \cdot) = dg(\cdot)$ . In general, a Poisson algebra is an associative  $k$ -algebra with a bilinear product  $\{ \cdot, \cdot \}$  that forms a Lie algebra (antisymmetric & Jacobi identity), and acts as a

derivation of the associative product, i.e.  $\{x, yz\} = \{x, y\}z + y\{x, z\}$ . Note that the commutator product of linear transformations satisfies these properties as well.

The Poisson bracket expresses the correspondence of Noether's theorem, between symmetries  $X_g$  and conserved quantities  $g$ . A symmetry is a vector field such that  $\omega$  and  $H$  are invariant under its flow. A conserved quantity is a function that is invariant under the flow of  $X_H$ , i.e. under time translation. Noether's theorem states that there is a 1-1 correspondence, under which conserved quantities  $f$  correspond to symmetries  $X_f$ , and the Poisson bracket of quantities corresponds to the commutator product of the corresponding symmetries, by

$$X_{\{f, g\}} = -[X_f, X_g].$$

In particular, given  $f(p, q)$ , compute

$$\begin{aligned} \dot{f}(p, q) &= \sum_i \left( \frac{\partial f}{\partial q^i} \dot{q}^i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) \\ &= \sum_i \left( \frac{\partial f}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q^i} \right) = \{f, H\}, \end{aligned}$$

so that if  $f$  is a conserved quantity,  $\{f, H\} = \dot{f} = 0$ . The symplectic structure is revealed in coordinates:

$$\{q^i, q^j\} = 0 = \{p_i, p_j\}, \{q^i, p_j\} = \delta_j^i.$$

## 5 Canonical Quantization

Another good reference for this is [3]. Each dynamical variable  $f \rightarrow \mathcal{O}_f$ , i.e. function on phase space, becomes a Hermitian operator in a fixed Hilbert space, the quantum analogue of phase space. Corresponding to the Poisson brackets is a commutator structure  $[A, B] = AB - BA$  on the algebra of such operators. You can verify that the commutator of two Hermitian operators is skew-Hermitian, i.e. that it is  $i$  times an Hermitian operator. This becomes

$$\{f, g\} \rightarrow -\frac{i}{\hbar} [\mathcal{O}_f, \mathcal{O}_g].$$

Corresponding to the position and momentum functions  $q, p$  are operators  $\mathcal{O}_q, \mathcal{O}_p$ , and from the definition of Poisson bracket we see that  $\{q, p\} = 1$ . Thus we insist that  $-\frac{i}{\hbar} [\mathcal{O}_q, \mathcal{O}_p] = 1$ . If we have the "Hilbert" space is  $C^\infty(\mathbb{R}, \mathbb{R})$ , then  $\mathcal{O}_p(f) = -i\hbar \frac{\partial f}{\partial x}$  and  $\mathcal{O}_q(f) = xf$ .

According to the Heisenberg picture, observable quantities are time-dependent Hermitian operators  $\mathcal{O}(t)$ , and the elements of the Hilbert space  $\psi$  are time-independent. Applying the quantization rule,

$$\frac{d}{dt} f = \{f, H\}$$

becomes

$$\frac{d}{dt} \mathcal{O}(t) = -\frac{i}{\hbar} [\mathcal{O}(t), \mathcal{O}_H], \text{ i.e. } \frac{d}{dt} \cdot = -\frac{i}{\hbar} [\cdot, \mathcal{O}_H].$$

The presence of  $\hbar$  in the quantization rule can now be explained as the constant needed to convert between units of energy ( $\mathcal{O}_H$ ) and inverse units of time ( $\frac{d}{dt}$ ).

According to the Schrödinger picture, the opposite happens – states are time dependent, and we can formally solve the last equation by  $\mathcal{O}(t) = e^{i\mathcal{O}_H t/\hbar} \mathcal{O} e^{-i\mathcal{O}_H t/\hbar}$ , where  $\mathcal{O}$  is a time independent operator. If we define  $\psi(t) = e^{-i\mathcal{O}_H t/\hbar} \psi$ , then differentiating gives

$$\frac{d}{dt} \psi(t) = e^{-i\mathcal{O}_H t/\hbar} \left( -\frac{i}{\hbar} \mathcal{O}_H \psi \right) = -\frac{i}{\hbar} \mathcal{O}_H \psi(t).$$

The result

$$\frac{d}{dt} \psi(t) = -\frac{i}{\hbar} \mathcal{O}_H \psi(t)$$

is called the Schrödinger equation.

Note that the quantization transforms

$$\{q^i, q^j\} = 0 = \{p_i, p_j\}, \{q^i, p_j\} = \delta_j^i$$

into

$$[\mathcal{O}_{q^i}, \mathcal{O}_{q^j}] = 0 = [\mathcal{O}_{p_i}, \mathcal{O}_{p_j}], [\mathcal{O}_{q^i}, \mathcal{O}_{p_j}] = i\hbar \delta_j^i.$$

It is also worth mentioning that the position operator  $\mathcal{O}_{q^i}$  is multiplication by  $q^i$ , and the momentum operator  $\mathcal{O}_{p_i}$  is  $-i\hbar \frac{d}{dq^i}$ .

Quantum states are equivalence classes of unit vectors in the Hilbert space, whose kernel is multiplication by modulus 1 complex numbers. We denote such a state as  $|\psi\rangle$ , and its (Hermitian conjugate, i.e. conjugate transpose) dual vector is denoted  $\langle\psi|$ . This gives  $\langle\psi|\psi\rangle = \|\psi\|$ , and  $\langle\cdot|\cdot\rangle$  expresses the Hermitian inner product on the Hilbert space.

## 6 The Harmonic Oscillator

Suppose the Lagrangian is given by

$$L = \frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2 q^2.$$

The corresponding Euler-Lagrange equation  $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$  is

$$\ddot{q} + \omega^2 q = 0,$$

i.e. Hooke's law  $\ddot{x} + (k/m)x = 0$  with frequency  $\omega = \sqrt{k/m}$ . The general solution of this differential equation is

$$q(t) = \frac{1}{\sqrt{2\omega}} (a e^{-i\omega t} + a^\dagger e^{i\omega t}),$$

where the physicist's dagger means complex conjugation. Momentum is defined by  $p = \partial L / \partial \dot{q} = \partial(\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2) / \partial \dot{q} = \dot{q}$ , so the momentum is given explicitly by

$$p(t) = -i\sqrt{\omega/2} (a e^{-i\omega t} - a^\dagger e^{i\omega t}).$$

We compute

$$\begin{aligned}\omega q + ip &= \sqrt{2\omega} a e^{-i\omega t} \\ \omega q - ip &= \sqrt{2\omega} a^\dagger \omega e^{i\omega t},\end{aligned}$$

which gives us

$$\begin{aligned}a &= \frac{1}{\sqrt{2\omega}} e^{i\omega t} (\omega q + ip) \\ a^\dagger &= \frac{1}{\sqrt{2\omega}} e^{-i\omega t} (\omega q - ip).\end{aligned}$$

Since here  $p = \dot{q}$ , the Hamiltonian is defined

$$H = \dot{q}p - L = p^2 - \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 = \frac{1}{2}(p^2 + \omega^2 q^2).$$

Leaving the details as exercise, using the general solutions for  $q$  and  $p$  we get

$$H = \omega/2(aa^\dagger + a^\dagger a).$$

Recall that we quantize by replacing  $q$  and  $p$  with operators  $\mathcal{O}_q, \mathcal{O}_p$  where  $[\mathcal{O}_q, \mathcal{O}_p] = i$ . The  $a$  and  $a^\dagger$  become operators, and the expressions for them in terms of  $q$  and  $p$  can be restated as

$$[a, a^\dagger] = 1.$$

To see how this works, apply the Poisson bracket to  $a$  and  $a^\dagger$ , and then quantize. We can use this expression to give the normal ordered form of the Hamiltonian

$$H = \omega/2(aa^\dagger + a^\dagger a) = \omega/2(1 + 2a^\dagger a) = \omega(a^\dagger a + 1/2).$$

Speaking of the Hamiltonian, let's examine its eigenvalues to see what the possible energy states of the system are. Suppose  $|\psi\rangle$  is an eigenstate of  $H$  with eigenvalue  $E$ . If we look at the norm of  $a|\psi\rangle$ , we have

$$0 \leq \|a|\psi\rangle\|^2 = \langle a|\psi|, a|\psi\rangle = \langle \psi|a^\dagger a|\psi\rangle,$$

which can be restated in terms of the Hamiltonian as

$$0 \leq \langle \psi| \frac{H}{\omega} - \frac{1}{2} |\psi\rangle = \frac{E}{\omega} - \frac{1}{2},$$

using  $\langle \psi|H|\psi\rangle = \langle \psi|E|\psi\rangle = E \|\psi\|^2 = E$ . This gives  $E \geq \omega/2$ . Note that if we had retained  $\hbar$  in the notation, we would have  $E \geq \hbar\omega/2$ .

You can check that

$$\begin{aligned}[H, a] &= -\omega a \\ [H, a^\dagger] &= \omega a^\dagger.\end{aligned}$$

If  $a|\psi\rangle \neq 0$ , we have

$$\begin{aligned}Ha|\psi\rangle &= ([H, a] + aH)|\psi\rangle \\ &= (-\omega a + aE)|\psi\rangle \\ &= (E - \omega)a|\psi\rangle,\end{aligned}$$

and similarly

$$Ha^\dagger|\psi\rangle = (E + \omega)a^\dagger|\psi\rangle.$$

In quantum field theory,  $a$  and  $a^\dagger$  are called annihilation and creation operators, respectively. Since  $E \geq \omega/2$ , the process of continued application of  $a$  must eventually terminate, i.e. there is a ground state  $|0\rangle$  such that  $a|0\rangle = 0$ . Then we have  $H|0\rangle = (\omega/2)|0\rangle$  (since  $E = 0$  is the eigenvalue), and we can define an orthonormal basis of states  $|n\rangle$  for  $n \in \mathbb{N}$  by repeatedly using the creation operator:

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \text{ equivalently } |n+1\rangle = \frac{1}{\sqrt{n+1}}a^\dagger|n\rangle.$$

The original factor of  $\frac{1}{\sqrt{2\omega}}$  was used so that  $\langle m|n\rangle = \delta_{mn}$ .

Note that the *number* operator  $N = a^\dagger a$  reports the level of an eigenstate:  $(a^\dagger a)|0\rangle = a^\dagger 0 = 0|0\rangle$ , and if  $(a^\dagger a)|n\rangle = n|n\rangle$ , then we have

$$\begin{aligned} (a^\dagger a)|n+1\rangle &= (a^\dagger a)\frac{1}{\sqrt{(n+1)!}}a^\dagger(a^\dagger)^n|0\rangle \\ &= (a^\dagger a a^\dagger)\frac{1}{\sqrt{(n+1)!}}\sqrt{n!}|n\rangle \\ &= \frac{1}{\sqrt{n+1}}a^\dagger(1 + a^\dagger a)|n\rangle \\ &= \frac{1}{\sqrt{n+1}}a^\dagger(|n\rangle + n|n\rangle) \\ &= \sqrt{n+1}a^\dagger|n\rangle \\ &= (n+1)|n+1\rangle. \end{aligned}$$

The eigenstates of the number operator are the  $|n\rangle$ , which form an orthonormal basis of states. These correspond to the harmonic series of frequencies in the oscillator. Since the Hamiltonian is essentially a scaled version of the number operator, these eigenstates correspond to the different possible classical energy levels of the oscillator. The Hamiltonian is thought to generate time translation, so you can think of it as continually oscillating by applying  $a$  and  $a^\dagger$ , which moves each state up one level, and then back down. According to the spectral theorems of unitary geometry, the Hamiltonian will always be diagonalizable, and its eigenstates can be taken to form an orthonormal basis for the state space.

## References

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