1 Symmetry in Classical and Quantum Mechanics

1.1 Principle of Least Action

Assume that there are n dynamical variables, $q_i(t)$. Define the *action* as $S = \int L(\dot{q}_i, q_i)\, dt$, where $L(\dot{q}_i, q_i)$ is a function called the *lagrangian* for the system, which is at least quadratic in $\dot{q}_i$. The principle of least action (also called *Hamilton's principle*) dictates that $\dot{q}_i$ and q_i behave so as to minimize S. The requirement $\delta S = 0$ gives rise to the Euler-Lagrange equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,, \tag{1.1}$$

which comprises a set of n second-order differential equations which describe the dynamics of q.

1.2 Hamiltonian Formulation

It is possible to describe the same dynamics described by (1.1) in terms of $2n$ first-order diferential equations. This is done by defining *momenta*

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}\,. \tag{1.2}$$

Since L is at least quadratic in $\dot{q}$, we can solve (1.2) to express $\dot{q}_i$ as a function of q_i and p_i, so that $\dot{q}_i = \dot{q}_i(q,p)$. Furthermore, if we differentiate (1.2) with respect to time, we derive

$$\begin{aligned} \dot{p}_i &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \\ &= \frac{\partial L}{\partial q_i}\,, \end{aligned} \tag{1.3}$$

where we have passed from the first line to the second using the Euler-Lagrange equation (1.1). We then define a new function

$$H \equiv \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\,, \tag{1.4}$$

which is called the *Hamiltonian*. Equation (1.4) is an example of a Legendre transformation. Since we use (1.2) to express $\dot{q}_i$ as a function of q_i and p_i, the Hamiltonian is a function of q_i and p_i; that is, we can rewrite (1.4) as

$$H(q,p) = \sum_{i=1}^{n} \dot{q}_i(q,p)\, p_i - L(\dot{q}(q,p), q)\,. \tag{1.5}$$

Now, if we vary H we determine

$$
\begin{aligned}
\delta H &= \sum_{i=1}^{n}\Big(p_i\,\delta\dot q_i + \dot q_i\,\delta p_i - \big(\frac{\partial L}{\partial \dot q_i}\,\delta\dot q_i + \frac{\partial L}{\partial q_i}\,\delta q_i\,\big)\Big)\\
&= \sum_{i=1}^{n}\Big(\dot q_i\,\delta p_i - \frac{\partial L}{\partial q_i}\,\delta q_i\Big)\\
&= \sum_{i=1}^{n}\Big(\dot q_i\,\delta p_i - \dot p_i\,\delta q_i\Big)\,, \qquad (1.6)
\end{aligned}
$$

where we have passed from the first line to the second by using (1.2), so that the first and third terms cancel, and we have passed from the second line to the third by using (1.3). Now, since δp_i and δq_i are independent variations, (1.6) tells us that

$$
\begin{aligned}
\dot q_i &= \frac{\partial H}{\partial p_i}\\
\dot p_i &= -\frac{\partial H}{\partial q_i}\,, \qquad (1.7)
\end{aligned}
$$

which comprises n first-order differential equations, called *Hamilton's equations*. Note that (1.7) is completely equivalent to the Euler-Lagrange equation (1.1).

Example:

As an example, consider a system with two dynamical variables, given by the position x and the velocity $\dot x$ of a particle moving in one-dimension in a potential $V(x)$, as described by the lagrangian $L(x,\dot x) = \frac{1}{2}\dot x^2 - V(x)$. The Euler-Lagrange equation (1.1) is then given by $\ddot x + V'(x) = 0$. According to (1.2), we have momentum p given by $p = \dot x$, so that the Hamiltonian (1.4) is $H(x,p) = \frac{1}{2}p^2 + V(x)$. The Hamilton equations (1.7) are then given by $\dot x = p$ and $\dot p = -V'(x)$. Eliminating p from these two equations tells us that $\ddot x + V'(x) = 0$, which is the same as the Euler-Lagrange equations, as expected.

1.3 Poisson Brackets

Suppose we are given $q_i(t)$ which satisfies (1.1), or, equivalently, $q_i(t)$ and $p_i(t)$ which satisfy (1.7), where p_i is defined by (1.2). Then we can determine the behavior of any function $f = f(q,p)$, since

$$\begin{aligned}\dot{f}(q,p) &= \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial q_i}\, \dot{q}_i + \frac{\partial f}{\partial p_i}\, \dot{p}_i \Big) \\ &= \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \Big) \\ &= \{\, f(q,p)\, ,\, H(q,p)\, \}_{\rm PB} \end{aligned} \tag{1.8}$$

where, in general, the *Poisson bracket* [1] is defined over two functions $f(q,p)$ and $g(q,p)$ as

$$\{\, f\, ,\, g\, \}_{\rm PB} \equiv \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \Big)\, . \tag{1.9}$$

The Poisson bracket is a bilinear operation which is skew-symmetric (meaning that $\{f, g\}_{\rm PB} = -\{g, f\}_{\rm PB}$, so that $\{f, f\}_{\rm PB} = 0$), and it also satisfies a Jacobi identity. Notice that all of these properties are satisfied, as well, by the commutator operation on matrices.

Note that the first of the two equations in (1.7) is equivalent to $\dot{q}_i = \{q_i, H\}_{\rm PB}$ and the second is equivalent to $\dot{p}_i = \{p_i, H\}_{\rm PB}$. Thus, a more general statement of Hamilton's principle is given by

$$\frac{d}{dt} f(q,p) = \{\, f(q,p)\, ,\, H(q,p)\, \}_{\rm PB}\, . \tag{1.10}$$

A consequence of (1.10) is that quantities $\mathcal{O}(q,p)$ which are conserved, meaning $\dot{\mathcal{O}} = 0$, have vanishing Poisson bracket $\{\mathcal{O}, H\}_{\rm PB} = 0$. Alternatively, any quantity $\mathcal{O}$ for which $\{\mathcal{O}, H\}_{\rm PB} = 0$ is conserved, $\dot{\mathcal{O}} = 0$.

1.4 Invariant Charges

Consider a lagrangian which depends on a particle's position $x(t)$ and its velocity $\dot{x}(t)$,

$$L = L(x, \dot{x})\, . \tag{1.11}$$

The position $x(t)$ is determined by the Euler-Lagrange equation,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0\, . \tag{1.12}$$

[1] Note that the definition given in (1.9) for the Poisson bracket is the standard definition found in classical mechanics texts, such as [1, 2], and is defined such that $\{q, p\}_{\rm PB} = 1$. But in the standard field theory texts, such as [3, 4], the Poisson bracket is defined with the opposite sign, so that $\{q, p\}_{\rm PB} = -1$.

The classical theory has a continuous symmetry if, under an infinitesimal change $x \rightarrow x + \delta_\theta x$, the lagrangian changes by a total derivative,

$$L \longrightarrow L + \dot{K}\,. \tag{1.13}$$

We specialize to the case where θ does not depend on t, $\dot{\theta} = 0$. The "charge", defined by

$$\tilde{Q}(\theta) \equiv \frac{\partial L}{\partial \dot{x}}\,\delta_\theta x - K\,, \tag{1.14}$$

is invariant, ie: $d\,\tilde{Q}(\theta)/dt = 0$. This is straightforward to prove, by simply differentiating,

$$\begin{aligned} \frac{d}{dt}\,\tilde{Q}(\theta) &= \Big(\frac{d}{dt}\,\frac{\partial L}{\partial \dot{x}}\Big)\,\delta_\theta x + \frac{\partial L}{\partial \dot{x}}\,\frac{d}{dt}\,\delta_\theta x - \dot{K} \\ &= \frac{\partial L}{\partial x}\,\delta_\theta x + \frac{\partial L}{\partial \dot{x}}\,\delta_\theta \dot{x} - \delta_\theta L \\ &= \delta_\theta L - \delta_\theta L \\ &= 0\,, \end{aligned} \tag{1.15}$$

where, in passing from the first line to the second, we have used (1.12) and the fact that the operator δ_θ commutes with the operator d/dt. The parameter θ will, in general, be a matrix-valued zero-form which can be expressed as $\theta = \theta^a\,T_a$, where $\{T_a\}$ are anti-Hermitian generators of a group. These then satisfy the commutation relation $[\,T_a\,,\,T_b\,] = f_{abc}\,T_c$, where f_{abc} are the (real) structure constants of the group. Since we have specialized to the case where $\dot{\theta} = 0$, the parameter can be extracted from the charge as

$$\tilde{Q}(\theta) = -\theta^a\,Q_a\,. \tag{1.16}$$

This defines the parameter-independent charge Q_a, which is also conserved, since $d\tilde{Q}(\theta)/dt = 0$ implies that $dQ_a/dt = 0$ (because the constant parameter passes through the derivative and can therefore be scaled out of the equation).

An interesting result is that the Poisson brackets of the charges Q_a reflects the group structure of the corresponding generators T_a,

$$\{\,Q_a\,,\,Q_b\,\}_{\rm PB} = f_{abc}\,Q_c\,, \tag{1.17}$$

where f_{abc} are the same structure constants defined by

$$[\,T_a\,,\,T_b\,] = f_{abc}\,T_c\,. \tag{1.18}$$

This fact is provides the essential link between classical mechanics and quantum mechanics.

1.5 Angular Momentum

As an example, consider a free point particle moving in three spatial dimensions, as described by the lagrangian

$$L = \sum_{i=1}^{3} \tfrac{1}{2}\,\dot{x}_i^2\,. \tag{1.19}$$

This theory has a global rotational symmetry, under which $x_i \to \Omega_{ij}\,x_j$, where the rotation matrix is generated by $(T_k)_{ij} = \varepsilon_{ijk}$. Thus,

$$\begin{aligned} x_i \quad &\longrightarrow \quad (\,e^{\theta^k T_k}\,)_{ij}\,x_j \\ &= \quad x_i + \varepsilon_{ijk} x_j \theta^k + \cdots\,, \end{aligned} \tag{1.20}$$

so that, infinitesimally, $\delta_\theta x_i = \varepsilon_{ijk} x_j \theta^k$. Note that the group of rotations generated by T_i is $SO(3)$, and that the T_i satisfy the commutation relationship $[\,T_i\,,\,T_j\,] = \varepsilon_{ijk}\,T_k$.

If we define the conserved charge to be $\tilde{J}(\theta)$, then, using (1.14), it is simple to determine

$$\begin{aligned} \tilde{J}(\theta) \quad &\equiv \quad \frac{\partial L}{\partial \dot{x}_i}\,\delta_\theta x_i \\ &= \quad \dot{x}^i(\varepsilon_{ijk} x_j \theta^k) \\ &= \quad -\theta^k\,(\varepsilon_{kij} x_i \dot{x}_j) \\ &\equiv \quad -\theta^i\,J_i\,, \end{aligned} \tag{1.21}$$

where the last line serves as a definition of the parameter-independent charge, which we can read off as

$$J_i = \varepsilon_{ijk}\,x_j\,\dot{x}_k \tag{1.22}$$

The momentum conjugate to x_i is defined by $p_i = \partial L/\partial \dot{x}_i = \dot{x}_i$, and the Hamiltonian, determined by (1.4) is

$$H = \sum_{i=1}^{3} \tfrac{1}{2}\,p_i^2\,. \tag{1.23}$$

In this case, we can rewrite (1.22) as

$$J_i = \varepsilon_{ijk}\,x_j\,p_k\,, \tag{1.24}$$

or, in common vector notation, $\mathbf{J} = \mathbf{r} \times \mathbf{p}$, which is the angular momentum. We therefore see that the conservation of angular momentum, ie: $\dot{\mathbf{J}} = 0$ is a consequence of the rotational symmetry of the theory, shown in (1.20).

It is instructive to compute the Poisson bracket,

$$\begin{aligned}
\{ J_i \,,\, J_j \}_{\mathrm{PB}} &= \sum_{l=1}^{3} \Big(\frac{\partial J_i}{\partial x_l} \frac{\partial J_j}{\partial p_l} - \frac{\partial J_i}{\partial p_l} \frac{\partial J_j}{\partial x_l} \Big) \\
&= \sum_{l=1}^{3} (\, \varepsilon_{ilk}\, \varepsilon_{jpl} - \varepsilon_{ipl}\, \varepsilon_{jlk} \,) x_p\, p_k \\
&= (\, \varepsilon_{ik}{}^{l}\, \varepsilon_{jlp} + \varepsilon_{kj}{}^{l}\, \varepsilon_{ilp} \,) x_p\, p_k \\
&= (-\varepsilon_{ji}{}^{l}\, \varepsilon_{klp}\,)\, x_p\, p_k \\
&= \varepsilon_{ij}{}^{l}\, (\, \varepsilon_{lpk}\, x_p\, p_k) \\
&= \varepsilon_{ijl}\, J_l \,. \qquad (1.25)
\end{aligned}$$

Notice that the Poisson brackets of the J_i reflect the commutator structure of the T_i.

1.6 Canonical Quantization

We quantize a classical theory by representing each dynamical variable f by a Hermitian operator in a Hilbert space, $f \to \widehat{f}$, and by imposing a commutator structure on the set of such operators which respects the corresponding algebra reflected in the classical Poisson brackets. This is done as follows,

$$\{ f \,,\, g \}_{\mathrm{PB}} \longrightarrow -\frac{i}{\hbar} [\, \widehat{f} \,,\, \widehat{g}\,] \,. \qquad (1.26)$$

Because of this relationship the quantum operators satisfy the same algebra as the transformations generated by the corresponding classical variables. (The factor of i is necessary in the quantum theory because the commutator of two Hermitian operators is anti-Hermitian.)

If we choose f and g to be the position and momentum, respectively, of a particle, so that $f = x$ and $g = p$, we easily determine the Poisson bracket $\{x, p\}_{\mathrm{PB}} = 1$. The *quantized* theory then includes operators $\widehat{x}$ and $\widehat{p}$, which satisfy the commutation relationship $-i[\widehat{x}, \widehat{p}]/\hbar = 1$, which we can rewrite as

$$[\, \widehat{x} \,,\, \widehat{p}\,] = i\, \hbar \,. \qquad (1.27)$$

If the Hilbert space is described by the space of smooth, continuous functions $f(x)$ (ie: maps $\mathbb{R} \to C^{\infty}(\mathbb{R})$), then the momentum operator is represented by $\widehat{p} = -i\, \hbar\, \partial / \partial x$.

There are two equivalent formulations of quantum mechanics. In the first of these, called the Heisenberg picture, the time-evolution is expressed in terms of operator evolution. In the second formulation, called the Shrödinger picture, the operators are time-independent, and the time evolution is included in a the state vector, or wavefunction. The Heisenberg picture is more useful for describing field quantization.

1.7 The Heisenberg picture

Hamilton's equation (1.10) describes the time evolution of a quantity $\mathcal{O}$ in the classical theory. In the Heisenberg picture of quantum mechanics, an observable quantity is represented by a (time-dependent) Hermitian operator $\mathcal{O}_H(t)$ which acts on (time-independent) elements of the Hilbert space, which we generically denote ψ_H. In this case, we can apply the quantization rule (1.26) to (1.10) to describe the time evolution of $\mathcal{O}_H(t)$ in the quantum theory. We thus determine

$$\frac{d}{dt}\,\widehat{\mathcal{O}}_H(t) = \frac{i}{\hbar}\,[\,\widehat{H}\,,\,\widehat{\mathcal{O}}_H(t)\,]\,. \tag{1.28}$$

Equation (1.28) describes the time-evolution of quantum operators in the Heisenberg picture.

1.8 The Schrödinger picture

Equation (1.28) can be formally solved by writing

$$\widehat{\mathcal{O}}_H(t) = e^{i\,\widehat{H}\,t/\hbar}\,\widehat{\mathcal{O}}_S\,e^{-i\,\widehat{H}\,t/\hbar}\,, \tag{1.29}$$

where $\widehat{\mathcal{O}}_S$ is a time-independent operator. Since the operator $\widehat{\mathcal{O}}_H$ acts on ψ_H, it is useful to define a time-dependent wavefunction as

$$\psi_S(t) \equiv e^{-i\,\widehat{H}\,t/\hbar}\,\psi_H \tag{1.30}$$

Differentiating this, we derive

$$\begin{aligned}
\frac{d}{dt}\,\psi_S(t) &= \frac{d}{dt}\,(\,e^{-i\,\widehat{H}\,t/\hbar}\,\psi_H\,) \\
&= e^{-i\,\widehat{H}\,t}\,(-\frac{i}{\hbar}\widehat{H}\,\psi_H) \\
&= -\frac{i}{\hbar}\widehat{H}\,(\,e^{-i\,\widehat{H}\,t}\,\psi_H\,) \\
&= -\frac{i}{\hbar}\widehat{H}\,\psi_S(t)
\end{aligned} \tag{1.31}$$

Thus, we determine that

$$\widehat{H}\,\psi_S(t) = i\,\hbar\,\frac{d}{dt}\,\psi_S(t)\,. \tag{1.32}$$

Equation (1.32) is called the Schrödinger equation. It is often convenient to work in units such that $\hbar = 1$.

1.9 Consistency

Consider the rotationally symmetric point-particle theory described by (1.19) and (1.23). The basic Poisson bracket is given by $\{x_i, p_j\}_{\rm PB} = \delta_{ij}$, which defines the canonical quantization condition given by

$$[\,\widehat{x}_i\,,\,\widehat{p}_j\,] = i\,\delta_{ij}\,. \tag{1.33}$$

The classically-conserved angular momentum is given by $J_i = \varepsilon_{ijk}\,x_j\,p_k$, and the associated Poisson bracket is $\{\,J_i\,,\,J_j\,\} = \varepsilon_{ijk}\,J_k$. So canonical quantization would imply that

$$[\,\widehat{J}_i\,,\,\widehat{J}_j\,] = i\,\varepsilon_{ijk}\,\widehat{J}_k\,. \tag{1.34}$$

In this case, if we simply take $\widehat{J}_i = \varepsilon_{ijk}\,\widehat{x}_j\,\widehat{p}_k$, and use the facts that the $\widehat{x}_i$ commute amongst each other, the $\widehat{p}_i$ commute amongst each other, and use (1.33), then a short computation shows that (1.34) follows automatically. Note that this simple consistency test was not guaranteed to hold on the basis of the tenets set forth above. It is good that it does hold because this enables us to consistently persue quantum mechanics along the simple lines addressed so far. But this is a good place to point out that consistency tests of precisely this sort do fail in certain field theories, indicating a deep relationship between symmetry principles and quantum mechanics. These are referred to as quantum mechanical anomalies.

1.10 Harmonic Oscillator

Consider the following classical lagrangian,

$$L = \tfrac{1}{2}\,\dot{q}^2 - \tfrac{1}{2}\,\omega^2\,q^2\,, \tag{1.35}$$

involving the single degree-of freedom $q(t)$. The Euler-Lagrange equation, obtained by extremizing (1.35), is given by

$$\ddot{q} + \omega^2\,q = 0\,. \tag{1.36}$$

The general soution to (1.36) is described by

$$q = \frac{1}{\sqrt{2\omega}} \left(a\, e^{-i\omega t} + a^\dagger\, e^{i\omega t} \right), \tag{1.37}$$

where a is an arbitrary real coefficient. The numerical prefactor in (1.37) has been chosen for future convenience. The momentum conjugate to q is defined by $p \equiv \partial L/\partial \dot{q}$, so that $p = \dot{q}$. Using the classical solution (1.37), the clasical momentum is described by

$$p = -i\sqrt{\frac{\omega}{2}} \left(a\, e^{-i\omega t} - a^\dagger\, e^{i\omega t} \right). \tag{1.38}$$

It is useful to use (1.37) and (1.38) to express a and $a^\dagger$ in terms of p and q, as follows,

$$\begin{aligned} a &= \frac{1}{\sqrt{2\omega}}\, e^{-i\omega t} \left(\omega q + ip \right) \\ a^\dagger &= \frac{1}{\sqrt{2\omega}}\, e^{i\omega t} \left(\omega q - ip \right) \end{aligned} \tag{1.39}$$

The Hamiltonian is defined by $H = p\dot{q} - L$, This tells us that $p = \dot{q}$ and

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\,\omega^2\, q^2\,. \tag{1.40}$$

Substituting the classsical results (1.37) and (1.38), we find

$$H = \frac{\omega}{2} \left(a\, a^\dagger + a^\dagger\, a \right). \tag{1.41}$$

We canonically quantize by replacing a and $a^\dagger$ with operators, subject to the condition $[\, q\,,\, p] = i$. Using (1.39), thius condition can be reexpressed as a condition on the *operators* a and $a^\dagger$, given by

$$[\, a\,,\, a^\dagger\,] = 1\,. \tag{1.42}$$

Using this result, we can express the hamiltonian (1.40) in *normal ordered* form as

$$H = \omega\, (a^\dagger\, a + \tfrac{1}{2}\,) \tag{1.43}$$

Define a "number" operator,

$$N \equiv a^\dagger\, a\,. \tag{1.44}$$

Using (1.42), this is easily shown to satisfy

$$\begin{aligned} [\, N\,,\, a^\dagger\,] &= a^\dagger \\ [\, N\,,\, a\,] &= -a\,. \end{aligned} \tag{1.45}$$

Assume that the number operator has eigenstates $|\,l>$, labeled by integer eigenvalues l. Thus, $N|\,l>= l\,|\,l>$. It is then easy to show that

$$
\begin{aligned}
N a\,|\,l> &= (aN - a)\,|\,l> = (l-1)\,a\,|\,l> \\
N a^{\dagger}\,|\,l> &= (a^{\dagger}N + a^{\dagger})\,|\,l> = (l+1)\,a^{\dagger}\,|\,l> \;.
\end{aligned}
\tag{1.46}
$$

Thus, $a\,|\,l> \propto |\,l-1>$, and $a^{\dagger}\,|\,l> \propto |\,l+1>$. For this reason, a is referred to as a "raising" operator, and $a^{\dagger}$ is called a "lowering" operator. The Hermitian conjugate of the state $|\,l>$ is denoted $<l\,|$. A positive definite norm is defined on the Hilbert space, so that

$$
< l\,|\,a^{\dagger}\,a\,|\,l> \;\geq 0\,. \tag{1.47}
$$

For this reason, there exists a "lowest" state $|\,0>$, normalized so that $<0\,|\,0>=1$, which is annihilated by the lowering operator,

$$
a\,|\,0>=0\,. \tag{1.48}
$$

This enables us to define an orthonormal basis of states

$$
|n\;>\equiv \frac{1}{\sqrt{n!}}\,a^{\dagger\,n}\,|\,0>\,, \tag{1.49}
$$

for $n = 0, 1, ..., \infty$, where the prefactor has been chosen so that $< m\,|\,n >= \delta_{mn}$.

References

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2 Symmetry in Classical Field Theory

A field theory involves separate dynamical variables at each point on a base manifold. In these lectures, the base manifold is flat four-dimensional Minkowski space. The simplest example involves a scalar field $\phi(x)$. A classical field theory is described by a *lagrangian density* $\mathcal{L} = \mathcal{L}(\,\phi, \partial_\mu \phi\,)$, so that the action is given by $S = \int d^4x\,\mathcal{L}$. The principle of least action dictates that $\phi(x)$ behaves so as to extremize S. The requirement $\delta S = 0$ gives rise to the Euler-Lagrange field equation,

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi(x)} - \frac{\delta \mathcal{L}}{\delta \phi(x)} = 0\,, \tag{2.1}$$

which is a second-order differential equation which describes the behavior of $\phi(x)$. As in the case of the classical mechanics of point particles, we can rephrase (2.1) in terms of first-order differential equations by performing a Legendre transformation. However, for the case of fields, this can be done only if we make a special distinction between the "space" and "time" coordinates.

To do this, we define the *momentum conjugate to* $\phi(x)$ as

$$\pi(t, \mathbf{x}) = \frac{\delta \mathcal{L}}{\delta \dot{\phi}(t, \mathbf{x})}\,, \tag{2.2}$$

where the dot denotes ∂_0. We can solve (2.1) to express $\dot{\phi}$ in terms of ϕ and π. Note that this places a special distinction on the time coordinate. Therefore, it is useful to rewrite the Euler Lagrange equation (2.1) as

$$\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\phi}(\mathbf{x}, t)} + \partial_i \frac{\delta \mathcal{L}}{\delta \partial_i \phi(\mathbf{x}, t)} - \frac{\delta \mathcal{L}}{\delta \phi(\mathbf{x}, t)} = 0\,, \tag{2.3}$$

where we recognize the first term as the time derivative of (2.2). Thus,

$$\dot{\pi}(\mathbf{x}, t) = \frac{\delta \mathcal{L}}{\delta \phi(\mathbf{x}, t)} - \partial_i \frac{\delta \mathcal{L}}{\delta \partial_i \phi(\mathbf{x}, t)}\,. \tag{2.4}$$

We now define a *Hamiltonian density* as

$$\mathcal{H} \equiv \dot{\phi}\, \pi - \mathcal{L}\,. \tag{2.5}$$

If we vary $\mathcal{H}$, we determine

$$\begin{aligned}
\delta \mathcal{H} &= \pi\, \delta \dot{\phi} + \dot{\phi}\, \delta \pi - \left(\frac{\delta \mathcal{L}}{\delta \phi}\, \delta \phi + \frac{\delta \mathcal{L}}{\delta \dot{\phi}}\, \delta\, \dot{\phi} + \frac{\delta \mathcal{L}}{\delta \partial_i \phi}\, \delta \partial_i \phi \right) \\
&= \dot{\phi}\, \delta \pi - \left(\dot{\pi} + \partial_i \frac{\delta \mathcal{L}}{\delta \partial_i \phi}\right) \delta \phi - \frac{\delta \mathcal{L}}{\delta \partial_i \phi}\, \partial_i\, \delta \phi \\
&= \dot{\phi}\, \delta \pi - \dot{\pi}\, \delta \phi - \partial_i \left(\frac{\delta \mathcal{L}}{\delta \partial_i \phi}\, \delta \phi \right),
\end{aligned} \tag{2.6}$$

where we have passed from the first line to the second by using (2.2) so that the first and fourth terms cancel, we have used (2.4) to rewrite the third term, and have used the fact that the infinitesimal field variation δ commutes with the derivative ∂_i. We now define the *Hamiltonian* to be the spatial integral of the Hamiltonian density,

$$H = \int d^3\mathbf{x}\, \mathcal{H}\,. \tag{2.7}$$

If we vary H, using (2.6) we determine

$$\delta\, H = \int d^3\mathbf{x}\, (\, \dot{\phi}\, \delta\pi - \dot{\pi}\, \delta\phi\,)\,. \tag{2.8}$$

The last term in (2.6) does not contribute to (2.8) because it is a total spatial derivative and therefore vanishes because of Stoke's theorem when integrated over (provided the fields vanish quickly enough as they approach infinity). But, since

$$\delta H = \int d^3\mathbf{x} \left(\frac{\delta H(t)}{\delta\phi(\mathbf{x},t)}\, \delta\phi(\mathbf{x},t) + \frac{\delta H(t)}{\delta\pi(\mathbf{x},t)}\, \delta\pi(\mathbf{x},t) \right), \tag{2.9}$$

where all functional variations are at *fixed times*, we have determined that

$$\begin{aligned} \dot{\phi}(\mathbf{x},t) &= \frac{\delta H(t)}{\delta\pi(\mathbf{x},t)} \\ \dot{\pi}(\mathbf{x},t) &= -\frac{\delta H(t)}{\delta\phi(\mathbf{x},t)}\,. \end{aligned} \tag{2.10}$$

In these expressions, we have indicated a formal time dependence in $H(t)$ in order to make the fixed-time used in the functional variation explicit; as it turns out, H is time-independent since it described the total energy. Equations (2.10) are field-version of Hamilton's equations, and are completely equivalent to the Euler-Lagrange field equation (2.1). We can use (2.10) to determine the dynamics of a generic functional $\mathcal{O} = \mathcal{O}(\, \phi(x)\, ,\, \pi(x)\, ,\, t\,)$, where, in addition to a field-dependence, we have included the possibility that the functional can have an additional explicit time-dependence. We find

$$\begin{aligned} \frac{d}{dt}\mathcal{O} &= \int d^3\mathbf{x} \left(\frac{\delta\mathcal{O}}{\delta\phi(\mathbf{x},t)}\, \dot{\phi}(\mathbf{x},t) + \frac{\delta\mathcal{O}}{\delta\pi(\mathbf{x},t)}\, \dot{\pi}(\mathbf{x},t) \right) + \frac{\partial\mathcal{O}}{\partial t} \\ &= \int d^3\mathbf{x} \left(\frac{\delta\mathcal{O}}{\delta\phi(\mathbf{x},t)}\, \frac{\delta H}{\delta\pi(\mathbf{x},t)} - \frac{\delta\mathcal{O}}{\delta\pi(\mathbf{x},t)}\, \frac{\delta H}{\delta\phi(\mathbf{x},t)} \right) + \frac{\partial\mathcal{O}}{\partial t} \\ &= \{\, \mathcal{O}\, ,\, H\, \}_{\mathrm{PB}} + \frac{\partial\mathcal{O}}{\partial t}\,, \end{aligned} \tag{2.11}$$

where the Poisson bracket is defined on any two functionals $F_1[\phi(x), \pi(x)]$ and $F_2[\phi(x), \pi(x)]$ as

$$\{\, F_1\, ,\, F_2\, \}_{\mathrm{PB}} \equiv \int d^3\mathbf{x} \left(\frac{\delta F_1}{\delta\phi(\mathbf{x},t)}\, \frac{\delta F_2}{\delta\pi(\mathbf{x},t)} - \frac{\delta F_1}{\delta\pi(\mathbf{x},t)}\, \frac{\delta F_2}{\delta\phi(\mathbf{x},t)} \right) \tag{2.12}$$

As in the case of point particle mechanics, the Poisson bracket is a bilinear operation which is skew-symmetric (meaning that $\{F_1, F_2\}_{\mathrm{PB}} = -\{F_2, F_1\}_{\mathrm{PB}}$, so that $\{F, F\} = 0$), and it also satisfies a Jacobi identity. Notice again that all of these properties are satisfied as well by the commutator operation on matrices. The most fundamental Poisson bracket is determined as follows,

$$\begin{aligned}\{\,\phi(\mathbf{x},t)\,,\,\pi(\mathbf{y},t)\,\} &= \int d^3\mathbf{w}\,\Big(\frac{\delta\phi(\mathbf{x},t)}{\delta\phi(\mathbf{w},t)}\frac{\delta\pi(\mathbf{y},t)}{\delta\pi(\mathbf{w},t)} - \frac{\delta\pi(\mathbf{x},t)}{\delta\phi(\mathbf{w},t)}\frac{\delta\phi(\mathbf{y},t)}{\delta\pi(\mathbf{w},t)}\Big) \\ &= \int d^2\mathbf{w}\,\delta^3(\mathbf{x}-\mathbf{w})\,\delta^3(\mathbf{y}-\mathbf{w}) \\ &= \delta^3(\mathbf{x}-\mathbf{y})\,, \end{aligned} \tag{2.13}$$

where the second term on the right-hand side of the first line vanishes because ϕ and π are a-priori independent, and the rest of the computation is a trivial exercize in functional calculus.

Equation (2.11) tells us that the Hamiltonian is responsible for the time translation of any quantity,

$$\frac{d}{dt}\,\mathcal{O} = \{\,\mathcal{O}\,,\,H\,\}_{\mathrm{PB}} + \frac{\partial\mathcal{O}}{\partial t}\,. \tag{2.14}$$

Note that this relation encompasses the Hamilton equations (2.10), since by choosing $\mathcal{O}=\phi$ or $\mathcal{O}=\pi$ we recover the first or second equation in (2.10), respectively. An important consequence of (2.14) is that any quantity whose Poisson bracket with the Hamiltonian vanishes is conserved. That is, if $\{F,H\}=0$, then $\dot{F}=0$. Note especially that $\{H,H\}=0$, so the Hamiltonian itself is conserved.

2.1 Conservation Laws

For every continous global symmetry transformation which leaves the action invariant, there is an associated current J^μ which is conserved (i.e. $\partial_\mu J^\mu = 0$), and a charge $Q=\int d^3\mathbf{x}\,J^0$ which is time-invariant (i.e. $\dot{Q}=0$). We can separate the symmetries of the theory into two classes. The first of these comprise the spacetime symmetries, consisting of the group of spacetime translations and the group of Lorentz transformations (i.e. spatial rotations plus boosts), which collectively describe the Poincaré group. The second class of symmetry transformations are the internal symmetries.

Translations:

Under a global translation $x^\mu \to x^\mu + \epsilon^\mu$, a scalar $\phi(x)$ transforms as $\phi(x)\to\phi(x+\epsilon) = \phi(x)+\epsilon^\mu\partial_\mu\phi(x)+\cdots$, so that $\delta\phi(x)=\epsilon^\mu\partial_\mu\phi(x)$. The lagrangian density, for instance, is a scalar, so it follows that

$$\delta\mathcal{L} = \epsilon^\mu\partial_\mu\mathcal{L}\,. \tag{2.15}$$

But, since $\mathcal{L} = \mathcal{L}(\,\phi(x)\,,\,\partial_\mu\phi(x)\,)$, it follows that

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi}\,\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\,\delta(\partial_\mu\phi) \\
&= \frac{\delta\mathcal{L}}{\delta\phi}\,\epsilon^\nu\partial_\nu\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\,\partial_\mu(\epsilon^\nu\partial_\nu\phi) \\
&= \epsilon^\nu\Big(\frac{\delta\mathcal{L}}{\delta\phi}\,\partial_\nu\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\,\partial_\mu\partial_\nu\phi\Big) \\
&= \epsilon^\nu\,\Big(\,\partial_\mu(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi})\partial_\nu\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\mu\partial_\nu\phi\Big) \\
&= \epsilon^\nu\partial_\mu\,(\,\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\nu\phi\,)\,,
\end{aligned} \tag{2.16}$$

where we pass from the second line to the third using the restriction that $\epsilon^\mu \neq \epsilon^\mu(x)$ and we pass from the third line to the fourth using the Euler-Lagrange field equation (2.1). Subtracting (2.16) from (2.15) we therefore determine

$$\epsilon^\nu\,\partial_\mu\,(\,\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\nu\phi - \delta^\nu{}_\mu\mathcal{L}\,) = 0\,. \tag{2.17}$$

We give the quantity in brackets a special name,

$$T^{\mu\nu} \equiv \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\,\partial^\nu\phi - g^{\mu\nu}\mathcal{L}\,, \tag{2.18}$$

which is called the *stress-energy tensor*. Since (2.17) is valid for *any* choice of ϵ^ν, it follows that (2.17) implies $\partial_\mu T^{\mu\nu} = 0$.

Lorentz transformations:

Under a global Lorentz transformation $x^\mu \to x^\mu + \varepsilon^{\mu\nu}x_\nu$, where $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$. A lorentz scalar $\phi(x)$ transforms as $\phi(x^\mu) \to \phi(x^\mu + \varepsilon^{\mu\nu}x_\nu) = \phi(x) + \varepsilon^{\mu\nu}x_\nu\partial_\mu\phi(x) + \cdots$, so that $\delta\phi = \varepsilon^{\mu\nu}x_\nu\partial_\mu\phi$. The lagrangian density, for instance, is a lorentz scalar, so it follows that

$$\begin{aligned}
\mathcal{L} &= \varepsilon^{\mu\rho}x_\rho\partial_\mu\mathcal{L} \\
&= \varepsilon^{\mu\rho}\partial_\mu(x_\rho\mathcal{L}) \\
&= \varepsilon^{\nu\rho}\partial_\mu(x_\rho\delta_\nu{}^\mu\mathcal{L})\,,
\end{aligned} \tag{2.19}$$

where we pass from the first line to the second using the observation that $\varepsilon^{\mu\nu}\partial_\mu x_\nu = \varepsilon^{\mu\nu}\eta_{\mu\nu} = 0$ (since $\varepsilon^{\mu\nu}$ is an antisymmetric tensor while $\eta_{\mu\nu}$ is a symmetric tensor), and we rewrite the result as shown in the third line for convenience below. But, since $\mathcal{L} =$

$\mathcal{L}(\,\phi(x)\,,\,\partial_\mu\phi(x)\,)$, it follows that

$$\begin{aligned}
\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi}\,\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\,\delta(\partial_\mu\phi) \\
&= (\,\partial_\mu\,\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\,)\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\,\partial_\mu(\delta\phi) \\
&= \partial_\mu\,(\,\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi\,) \\
&= \partial_\mu\,(\,\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\,\varepsilon^{\nu\rho}x_\rho\partial_\nu\phi\,) \\
&= \varepsilon^{\nu\rho}\partial_\mu\,(\,x_\rho\,\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\,\partial_\nu\phi\,)\,,
\end{aligned} \tag{2.20}$$

where we have passed from the first line to the second using the Euler-Lagrange field equation (2.1), and we pass to the final line by using the restriction that $\varepsilon^{\mu\nu} \neq \varepsilon^{\mu\nu}(x)$. Subtracting (2.19) from (2.20) we determine

$$\begin{aligned}
0 &= \varepsilon^{\nu\rho}\partial_\mu\Big(\,(\,\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\,\partial_\nu\phi - \delta_\nu{}^\mu\,\mathcal{L}\,)x_\rho\Big) \\
&= \varepsilon_{\nu\rho}\,\partial_\mu\Big(T^{\mu\nu}\,x^\rho\Big) \\
&= \tfrac{1}{2}\,\varepsilon_{\nu\rho}\,\partial_\mu\Big(T^{\mu\nu}\,x^\rho - T^{\mu\rho}\,x^\nu\Big)\,.
\end{aligned} \tag{2.21}$$

We give the quantity in brackets a special name,

$$M^{\mu\nu\rho} = T^{\mu\nu}\,x^\rho - T^{\mu\rho}\,x^\nu\,. \tag{2.22}$$

Since (2.21) is valid for any choice of $\varepsilon^{\mu\nu}$ it follows that (2.21) implies that $\partial_\mu\,M^{\mu\nu\rho} = 0$.

2.2 The Poincaré Group

The Poincaré group consists of four spacetime translations, generated by P_μ plus the Lorentz group, which involves the six generators $M_{\mu\nu}$, which are antisymmetric in the indices. These satisfy the commutation relationships

$$\begin{aligned}
[\,M_{\mu\nu}\,,\,M_{\lambda\sigma}\,] &= \eta_{\mu\lambda}\,M_{\nu\sigma} + \eta_{\nu\sigma}\,M_{\mu\lambda} - \eta_{\mu\sigma}\,M_{\nu\lambda} - \eta_{\nu\lambda}\,M_{\mu\sigma} \\
[\,M_{\mu\nu}\,,\,P_\lambda\,] &= \eta_{\mu\lambda}\,P_\nu - \eta_{\nu\lambda}\,P_\mu\,,
\end{aligned} \tag{2.23}$$

where $\mu = 0,1,2,3$ and $\eta_{\mu\nu} = \mathrm{diag}(+---)$. It is instructive to separate those generators involving a time index from the others. Thus, the four spacetime translations P_μ separate into $H \equiv P_0$, which describes time translations, and P_i, where $i = 1, 2, 3$, whch describes spatial translations. Similarly, the six generators $M_{\mu\nu}$ separate into $X_i \equiv M_{i0}$, which describe boosts in each of the three spatial directions, and $L_i \equiv \frac{1}{2}\varepsilon_{ijk} M_{jk}$, where ε_{ijk} is the totally antisymmetric symbol with $\varepsilon_{123} = 1$, which describes the three spatial rotations. The algebra (2.23) can then be rewritten as follows, where all omitted commutators vanish,

$$\begin{aligned} [\, L_i \, , \, L_j \,] &= \varepsilon_{ijk} \, L_k & \qquad [\, H \, , \, X_i \,] &= P_i \\ [\, X_i \, , \, L_j \,] &= \varepsilon_{ijk} \, X_k & \qquad [\, P_i \, , \, X_j \,] &= \delta_{ij} \, H \\ [\, X_i \, , \, X_j \,] &= \varepsilon_{ijk} \, L_k & \qquad [\, P_i \, , \, L_j \,] &= \varepsilon_{ijk} \, P_k \, . \end{aligned} \tag{2.24}$$

Note that the only generators which do not commute with time translation, H, are the boosts, X_i.

Example:

The simplest example of a field theory in $d = 4$ Minkowski space is the theory of a free real scalar field $\phi(x)$, described by the lagrangian density

$$\mathcal{L} = \tfrac{1}{2} \, \partial_\mu \phi \, \partial^\mu \phi \, . \tag{2.25}$$

It is straightforward to determine the two conserved currents $T^{\mu\nu}$ and $M^{\mu\nu\lambda}$, using (2.18) and (2.22),

$$\begin{aligned} T^{\mu\nu} &= \partial^\mu \phi \, \partial^\nu \phi - \tfrac{1}{2} \, g^{\mu\nu} \, \partial_\rho \phi \, \partial^\rho \phi \\ M^{\mu\nu\lambda} &= T^{\mu\nu} \, x^\lambda - T^{\mu\lambda} \, x^\nu \end{aligned} \tag{2.26}$$

The momentum conjugate to ϕ is defined by (2.2) to be $\pi(x) = \dot{\phi}(x)$. Therefore, if we make a special distinction for the time direction, and define the individual invariant charges as

$$\begin{aligned} H &= \int d^3\mathbf{x} \, T_{00} & \qquad X_i &= \int d^3\mathbf{x} \, M_{0i0} \\ P_i &= \int d^3\mathbf{x} \, T_{0i} & \qquad L_i &= \tfrac{1}{2} \, \varepsilon_{ijk} \int d^3\mathbf{x} \, M^{0jk} \, , \end{aligned} \tag{2.27}$$

we determine the following explicit expressions for the charges,

$$\begin{aligned} H &= \int d^3\mathbf{x} \left(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}\,(\partial_i\phi)^2 \right) \\ P_i &= \int d^3\mathbf{x}\, \pi\, \partial_i\phi \\ X_i &= \int d^3\mathbf{x} \left(\tfrac{1}{2}\, x^i\, \pi^2 + t\, \pi\, \partial_i\phi + \tfrac{1}{2}\, x^i\, (\partial_j\phi)^2 \right) \\ L_i &= \varepsilon_{ijk} \int d^3\mathbf{x}\, \pi\, x^j\, \partial_k\phi\,. \end{aligned} \tag{2.28}$$

Note that only the X_i have an explicit time-dependence. Now we can compute the Poisson bracket (2.12) between any pair of these expressions. After this straightforward exercise, we determine

$$\begin{aligned} \{\, L_i\,,\, L_j\,\}_{\mathrm{PB}} &= \varepsilon_{ijk}\, L_k \qquad & \{\, H\,,\, X_i\,\}_{\mathrm{PB}} = P_i \\ \{\, X_i\,,\, L_j\,\}_{\mathrm{PB}} &= \varepsilon_{ijk}\, X_k \qquad & \{\, P_i\,,\, X_j\,\}_{\mathrm{PB}} = \delta_{ij}\, H \\ \{\, X_i\,,\, X_j\,\}_{\mathrm{PB}} &= \varepsilon_{ijk}\, L_k \qquad & \{\, P_i\,,\, L_j\,\}_{\mathrm{PB}} = \varepsilon_{ijk}\, P_k\,, \end{aligned} \tag{2.29}$$

which, as is easily verified, precisely reflects the Poincaré algebra (2.24). Using the functional Hamilton equations (2.14), we verify that each of these charges are invariant, ie: $\dot{H} = \dot{P}_i = \dot{L}_i = \dot{X}_i = 0$. This follows for H, P_i and L_i because each of these have vanishing Poisson bracket with H. For X_i it follows because of the explicit time dependence in the expression for X_i, from which it is easy to compute

$$\frac{\partial X_i}{\partial t} = P_i\,. \tag{2.30}$$

Thus, we determine

$$\begin{aligned} \frac{d}{dt} X_i &= \{\, X_i\,,\, H\,\}_{\mathrm{PB}} + \frac{\partial X_i}{\partial t} \\ &= -P_i + P_i \\ &= 0\,. \end{aligned} \tag{2.31}$$

So, we have verified, using the Hamiltonian approach, that there is, indeed, a conserved quantity associated with each of the parameters of the ten-dimensional Poincaré group.

2.3 Noether's Theorem

So far, we have demonstrated that, associated with each of the ten parameters associated with the Poincaré group, there exists a corresponding conserved current and invariant charge. This is an example of a generic phenomenon. In any classical field theory, associated with *each* independent continuous global symmetry transformation there is a corresponding conserved current and invariant charge. In this section, we describe the phenomenon in generality.

Consider a classical field theory described by a lagrangian density which depends on a field ϕ and its derivatives $\partial_\mu \phi$,

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)\,. \tag{2.32}$$

The field ϕ obeys the Euler-Lagrange equation of motion,

$$\partial_\mu\left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\right) - \frac{\delta \mathcal{L}}{\delta \phi} = 0\,. \tag{2.33}$$

The theory has a symmetry if, under the change $\phi \longrightarrow \phi + \delta_\theta \phi$ the lagrangian density changes by a total derivative,

$$\mathcal{L} \longrightarrow \mathcal{L} + \partial_\mu K^\mu\,. \tag{2.34}$$

We specialize to the case where θ is a spacetime constant, $\partial_\mu\, \theta = 0$. The current, defined by

$$\tilde{J}^\mu(\theta) \equiv \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\, \delta_\theta \phi - K^\mu\,, \tag{2.35}$$

is conserved, ie: $\partial_\mu \tilde{J}^\mu(\theta) = 0$. This is straightforward to prove, by simply differentiating,

$$\begin{aligned} \partial_\mu \tilde{J}^\mu(\theta) &= (\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi})\, \delta_\theta \phi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\, \partial_\mu(\delta_\theta \phi) - \partial_\mu K^\mu \\ &= \frac{\delta \mathcal{L}}{\delta \phi}\, \delta_\theta \phi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\, \delta_\theta(\partial_\mu \phi) - \delta_\theta \mathcal{L} \\ &= \delta_\theta \mathcal{L} - \delta_\theta \mathcal{L} \\ &= 0\,, \end{aligned} \tag{2.36}$$

where, in passing from the first line to the second, we have used (2.33) and the fact that the operator δ_θ commutes with the operator ∂_μ.

The parameter θ will, in general, be a (constant) matrix-valued zero-form which can be expressed as $\theta = \theta^a T_a$, where $\{T_a\}$ are the anti-Hermitian generators of a group. Therefore, the parameter can be extracted from the current as

$$\tilde{J}^\mu(\theta) = -\theta^a\, J^\mu_a\,. \tag{2.37}$$

This defines the parameter-independent current J^μ_a, which is also conserved, since the $\partial_\mu \tilde{J}^\mu(\theta) = 0$ implies that $\partial_\mu J^\mu_a = 0$ (because the constant parameter passes through the derivative and can be therefore be scaled out of the equation).

2.4 Example

As an example, consider the lagrangian density,

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi\,. \tag{2.38}$$

Vector symmetry:

Equation (2.38) is invariant under the global transformation

$$\begin{aligned} \psi &\longrightarrow g^{-1}\,\psi \\ \bar{\psi} &\longrightarrow \bar{\psi}\,g\,. \end{aligned} \tag{2.39}$$

The group element g is represented as $g = \exp\theta$, where $\theta = \theta^a T_a$ is a matrix-valued parameter, and $\{\theta^a\}$ are real numbers. In this case, the infinitesimal transformations are given by $\delta_\theta\psi = -\theta\,\psi$, $\delta_\theta\bar{\psi} = \bar{\psi}\,\theta$. It is easy to show that $\delta_\theta\mathcal{L} = 0$, so that $K^\mu = 0$. Furthermore, since there is only one derivative in the lagrangian density, the current is given by

$$\begin{aligned} \tilde{J}^\mu(\theta) &= \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\,\delta\psi \\ &= -\theta^a(\,i\,\bar{\psi}\gamma^\mu\,T_a\,\psi\,)\,. \end{aligned} \tag{2.40}$$

Therefore, the parameter-independent conserved current is given by

$$J^\mu_a = i\,\bar{\psi}\gamma^\mu\,T_a\,\psi\,, \tag{2.41}$$

and the corresponding conservation law tells us that $\partial_\mu J^\mu_a = 0$.

Axial Symmetry:

Equation (2.38) is also invariant under the second global transformation

$$\begin{aligned} \psi &\longrightarrow e^{i\alpha\gamma_5}\,\psi \\ \bar{\psi} &\longrightarrow \bar{\psi}\,e^{i\alpha\gamma_5}\,, \end{aligned} \tag{2.42}$$

where α is a real, constant angular variable. In this case, the infinitesimal transformations are given by $\delta_\alpha\psi = i\alpha\,\gamma_5\psi$ and $\delta_\alpha\bar{\psi} = i\alpha\bar{\psi}\,\gamma_5$. It is easy to show that $\delta_\alpha\mathcal{L} = 0$, so that $K^\mu = 0$. Furthermore, since there is only one derivative in the lagrangian density, the current is given by

$$\begin{aligned} \tilde{J}^\mu_5(\alpha) &= \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\,\delta_\alpha\psi \\ &= -\alpha(\,\bar{\psi}\gamma^\mu\gamma_5\,\psi\,)\,. \end{aligned} \tag{2.43}$$

Therefore, the parameter-independent conserved current is given by

$$J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\,\psi\,, \tag{2.44}$$

where the subscript 5 has been included in the name J_5^μ, due to the presence of the γ_5 factor, to distinguish this current from the vector current given in (2.41).

3 Quantization of Real Scalar Fields

We quantize a field theory involving a real scalar field $\phi(x)$, with conjugate momenta $\pi(x)$, by replacing $\phi(x)$ and $\pi(x)$ with Hermitian operators, $\widehat{\phi}(x)$ and $\widehat{\pi}(x)$, which satisfy operator commutation relations *at equal times*, which respects the corresponding algebra reflected in the classical Poisson brackets [2]. This is done as follows,

$$\{\,F_1\,,\,F_2\,\}_{\rm PB} \longrightarrow -\frac{i}{\hbar}\,[\,\widehat{F}_1\,,\,\widehat{F}_2\,]\,, \tag{3.1}$$

where F_1 and F_2 are generic functionals of $\widehat{\phi}(x)$ and $\widehat{\pi}(x)$. Because of relationship (3.1), the quantum operators corresponding to conserved charges should satisfy the same commutator algebra as the transformations generated by the corresponding symmetry transformations. The factor of i is necessary in the quantum theory because the commutator of two Hermitian operators is anti-Hermitian. If we apply (3.1) to the case where F_1 and F_2 are taken to $\phi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$ respectively, we detemine

$$[\,\widehat{\phi}(\mathbf{x},t)\,,\,\widehat{\pi}(\mathbf{y},t)\,] = i\,\hbar\,\delta^3(\mathbf{x}-\mathbf{y})\,. \tag{3.2}$$

This "equal time commutator" is a fundamental expression in the quantization of fields.

3.1 Free Real Scalar Field Theory

A free real scalar field is described by the following lagrangian density [3]

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}\,m^2\,\phi^2\,. \tag{3.3}$$

The Euler-Lagrange field equation, obtained by functionally extremizing the action $S = \int d^4x\,\mathcal{L}$, is given by

$$(\,\Box + m^2\,)\,\phi(x) = 0\,. \tag{3.4}$$

[2] In this lecture, all fields are assumed to be bosonic.

[3] Henceforth, we will use the more concise term *lagrangian* rather than *lagrangian density* when the distinction is clear.

This particular equation is called the Klein-Gordon equation. The general solution is given by,

$$\phi(x) = \int d^3\mathbf{k} \left(a(\mathbf{k})\, f_k(x) + a^\dagger(\mathbf{k})\, f_k^*(x) \right) \tag{3.5}$$

where $f_k(x)$ is an individual plane wave solution, given by

$$f_k(x) = \frac{1}{\sqrt{(2\pi)^3\, 2\omega_k}}\, e^{-i\,k\cdot x} \tag{3.6}$$

with energy ω_k, given by $\omega_k = \sqrt{\mathbf{k}\cdot\mathbf{k} + m^2}$. The normalization in (3.6) is chosen such that

$$\begin{aligned} \int d^3\mathbf{x}\, f_k^*(x)\, i \overleftrightarrow{\partial_0}\, f_{k'}(x) &= \delta^3(\,\mathbf{k} - \mathbf{k}'\,) \\ \int d^3\mathbf{x}\, f_k(x)\, i \overleftrightarrow{\partial_0}\, f_{k'}(x) &= 0\,. \end{aligned} \tag{3.7}$$

3.2 Quantize the theory

The canonical momentum associated with $\phi(x)$ is given by $\pi(x) = \delta\mathcal{L}/\delta\dot{\phi}(x) = \dot{\phi}(x)$. We quantize the theory by imposing the following equal-time commutation relationship,

$$[\,\pi(\mathbf{x}, t)\,,\, \phi(\mathbf{y}, t)\,] = -i\delta^3(\,\mathbf{x} - \mathbf{y}\,)\,. \tag{3.8}$$

In terms of the operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$, this relationship is rephrased as follows,

$$[\,a(\mathbf{k})\,,\, a^\dagger(\mathbf{k}')\,] = \delta^3(\mathbf{k} - \mathbf{k}'\,)\,, \tag{3.9}$$

which is practically identical to the harmonic oscillator algebra described previously. Thus, a quantized scalar field is essentially a collection of harmonic oscillators. Using reasoning similar to that discussed above, there exists a ground state $|\,0\rangle$, now called "the vacuum", which is annihilated by $a(\mathbf{k})$, ie: $a(\mathbf{k})\,|\,0\rangle = 0$, and which is normalized so that $\langle 0\,|\,0\rangle = 1$. Furthermore, there is a complete set of orthonormal basis states, which are constructed by acting with powers of $a^\dagger(\mathbf{k})$ on the vacuum.

3.3 Some Useful Functions

Of interest is the two-point function $\langle 0 \,|\, \phi(y)\, \phi(x) \,|\, 0\rangle$, which is also called a "correlation function". It is straightforward to compute this,

$$
\begin{aligned}
&\langle 0 \,|\, \phi(y)\, \phi(x) \,|\, 0\rangle \\
&\quad = \langle 0 \,| \int d^3\mathbf{k}\, d^3\mathbf{k}' \left(a(\mathbf{k}) f_k(y) + a^\dagger(\mathbf{k}) f_k^*(y) \right) \left(a(\mathbf{k}') f_k(x) + a^\dagger(\mathbf{k}') f_k^*(x) \right) |\, 0\rangle \\
&\quad = \int d^3\mathbf{k}\, d^3\mathbf{k}' \langle 0 \,|\, a(\mathbf{k}) a^\dagger(\mathbf{k}') \,|\, 0\rangle\, f_k(y) f_{k'}^*(x) \\
&\quad = \int d^3\mathbf{k}\, f_k(y) f_k^*(x) \\
&\quad = \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2\omega_k}\, e^{-i\,k\cdot(y-x)}\,. \qquad (3.10)
\end{aligned}
$$

In this calculation, we pass from the first line to the second using the facts that $a(\mathbf{k})|0 >= 0$, which defines the vacuum, and $\langle 0 \,|\, a^\dagger(\mathbf{k}) a^\dagger(\mathbf{k}') \,|0\rangle = 0$, which follows because the state $a^\dagger(\mathbf{k}) a^\dagger(\mathbf{k}')|\, 0\rangle$ is normal to the vacuum, and we pass from the second line to the third by substituting $\langle 0 \,|\, a(\mathbf{k}) a^\dagger(\mathbf{k}') \,|\, 0\rangle = \langle 0 \,|\, [\, a(\mathbf{k})\,,\, a^\dagger(\mathbf{k}')\,] \,|\, 0\rangle = \delta^3(\mathbf{k} - \mathbf{k}')$. It is interesting to note that (3.10) is Lorentz invariant. This follows because the integration measure is Lorentz invariant (*prove this*) and because the integrand is manifestly Lorentz invariant.

Another object of interest is the vacuum-to-vacuum amplitude of the commutator,

$$
\begin{aligned}
\Delta(x-y) &\equiv -i\, \langle 0 \,|\, [\, \phi(x)\,,\, \phi(y)\,] \,|\, 0\rangle \\
&= -i \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2\omega_k} \left(e^{-i\,k\cdot(x-y)} - e^{i\,k\cdot(x-y)} \right), \qquad (3.11)
\end{aligned}
$$

where the first identity *defines* $\Delta(x-y)$ and the second identity follows obviously from (3.10). The function $\Delta(x-y)$ has some useful properties, such as

$$
\begin{aligned}
(\Box_x^2 + m^2)\, \Delta(x-y) &= 0 \\
\Delta(x-y)\,|_{x_0=y_0} &= 0 \\
\partial_0^x\, \Delta(x-y)\,|_{x_0=y_0} &= -\delta^3(\mathbf{x} - \mathbf{y})\,. \qquad (3.12)
\end{aligned}
$$

The first of these relations is simple to determine, using the property $k^2 = m^2$, the second is straightforward to prove by replacing $\mathbf{k} \to -\mathbf{k}$ in the second integral, and the third is similarly straightforward, using the integral representation of the delta function described in the appendix. Furthermore $\Delta(x-y)$ is Lorentz invariant, just as (3.10) is Lorentz