Econ 300G **Paul Corcoro**

Math Review Practice Problems

I. Derivatives

A) Case of one variable: take derivatives of the following functions.

1. $\frac{dy}{dx} = 3$
2. $\frac{dy}{dx} = 12$
3. $\frac{dy}{dx} = 4 \cdot 3x^{3-1} = 12x^2$
4. $\frac{dy}{dx} = 10x$
5. $\frac{dy}{dx} = 2 \cdot \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{\sqrt{x}}$
6. $\frac{dy}{dx} = 60x^4 - 16x^3$
7. $\frac{dy}{dx} = \frac{1}{\sqrt{x}} + 7.5\sqrt{x}$
8. $\frac{dy}{dx} = 12x^3(2x-5) + 3x^4 \cdot 2 = 30x^4 - 60x^3$
9. $\frac{dy}{dx} = 2(3x^3 - 4x^2 + 6)(9x^2 - 8x)$
10. $\frac{dy}{dx} = 4(5x^2+3)^3 10x$
11. $\frac{dy}{dx} = \frac{1}{(x^2+3x-5)}(2x+3)$
12. $\frac{dy}{dx} = 2\ln(x)\frac{1}{x}$

B) Case of several variables: take partial derivatives of the following functions (i.e. determine f_x and f_z)

1. $\frac{\partial y}{\partial x} = 30x^2z^2 + 6z$ and $\frac{\partial y}{\partial z} = 20x^3z + 6x$
2. $\frac{\partial y}{\partial x} = 15x^2 - 6xz^2$ and $\frac{\partial y}{\partial z} = -6x^2z + 35z^4$
3. $\frac{\partial y}{\partial x} = 6xz^3$ and $\frac{\partial y}{\partial z} = 9x^2z^2$

4. $\frac{\partial y}{\partial x} = 10\sqrt{z} \cdot \frac{1}{2\sqrt{x}} = 5\sqrt{\frac{z}{x}}$ and $\frac{\partial y}{\partial z} = 10\sqrt{x}\,\frac{1}{2\sqrt{z}} = 5\sqrt{\frac{x}{z}}$
5. $\frac{\partial y}{\partial x} = 4x - 5z$ and $\frac{\partial y}{\partial z} = 5x - 6z$
6. $\frac{\partial y}{\partial x} = 2x + 3z$ and $\frac{\partial y}{\partial z} = 3x - 5$
7. $\frac{\partial y}{\partial x} = 3\left[\ln\left(x+z\right)\right]^2 \frac{1}{\left(x+z\right)}$ and $\frac{\partial y}{\partial z} = 3\left[\ln\left(x+z\right)\right]^2 \frac{1}{\left(x+z\right)} + 2x$

II. Unconstrained Optimization

A) Find values of x at which function y is optimized.

1. FOC: $\frac{dy}{dx} = -3x^2 + 12x + 15 = 0$

   $3x^2 - 12x - 15 = 0$

   $x^2 - 4x - 5 = 0$

   $x_1 = \frac{4 - \sqrt{16 + 4 \cdot 5}}{2} = -1$

   $x_2 = \frac{4 + \sqrt{16 + 4 \cdot 5}}{2} = 5$

2. FOC: $\frac{dy}{dx} = 2x - 7 = 0 \Rightarrow \boxed{x^* = 3.5}$

B) Find values of x and z at which function y is optimized.

1. FOCs:

   $$\frac{\partial y}{\partial x} = -6x - 6z + 72 = 0 \qquad (1)$$

   $$\frac{\partial y}{\partial z} = 48 - 6x - 4z = 0 \qquad (2)$$

   From (1) we get x as a function of z: $x = 12 - z$. We will use this function and substitute for x into (2) to get the following:

   $$48 - 6\left(12 - z\right) - 4z = 0 \;\Rightarrow\; 48 - 72 + 6z - 4z = 0 \;\Rightarrow\; 2z = 24 \;\Rightarrow\; \boxed{z^* = 12}$$

   from this we can solve for the optimal value of x: $\boxed{x^* = 12 - 12 = 0}$

2. FOCs:

   $$\frac{\partial y}{\partial x} = 6x - z - 4 = 0 \qquad (1)$$

$$\frac{\partial y}{\partial z} = -x + 4z - 7 = 0 \qquad (2)$$

From (1) we get $z = 6x - 4$. Substitute for z into (2) to get:

$$-x + 4(6x - 4) - 7 = 0 \Rightarrow -x + 24x - 16 - 7 = 0 \Rightarrow 23x = 23 \Rightarrow \boxed{x^* = 1}$$

and $\boxed{z^* = 6 - 4 = 2}$

III Constrained Optimization

Solve constrained optimization problems below (i.e. find values of x and z that optimize objective function y subject to the constraint). Calculate Lagrange Multiplier and interpret it.

Problem 1

$y = 4x^{\frac{1}{4}}z^{\frac{3}{4}}$

subject to

$100 = 2x + 6z$

$$L = 4x^{\frac{1}{4}}z^{\frac{3}{4}} + \lambda[100 - 2x - 6z]$$

FOCs:

$$\frac{\partial L}{\partial x} = 4 \cdot \frac{1}{4} x^{-\frac{3}{4}} z^{\frac{3}{4}} - 2\lambda = 0 \qquad (1)$$

$$\frac{\partial L}{\partial z} = 4 \cdot \frac{3}{4} x^{\frac{1}{4}} z^{-\frac{1}{4}} - 6\lambda = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = 100 - 2x - 6z = 0 \qquad (3)$$

Solve FOCs simultaneously for x^*, z^* and λ^*

From (1) $\lambda = \dfrac{z^{\frac{3}{4}}}{2x^{\frac{3}{4}}}$ plug into (2) to get the following:

$$\frac{3x^{\frac{1}{4}}}{z^{\frac{1}{4}}} - 6\left(\frac{z^{\frac{3}{4}}}{2x^{\frac{3}{4}}}\right) = 0 \Rightarrow \frac{x^{\frac{1}{4}}}{z^{\frac{1}{4}}} = \frac{z^{\frac{3}{4}}}{x^{\frac{3}{4}}} \Rightarrow x^{\frac{1}{4}} \cdot x^{\frac{3}{4}} = z^{\frac{1}{4}} \cdot z^{\frac{3}{4}} \Rightarrow \qquad x = z$$

using this result (3) becomes $2x + 6x = 100 \Rightarrow$ $\boxed{x^* = z^* = 12.5}$

$\boxed{\lambda^* = 0.5}$ and it implies that if a constant of the constraint increases by one unit, the value of the objective function y would approximately increase by 0.5.

Problem 2

$y = 26x - 3x^2 + 5xz - 6z^2 + 12z$
subject to
$170 = 3x + z$

$L = 26x - 3x^2 + 5xz - 6z^2 + 12z + \lambda[170 - 3x - z]$

FOCs:

$$\frac{\partial L}{\partial x} = 26 - 6x + 5z - 3\lambda = 0 \qquad (1)$$

$$\frac{\partial L}{\partial z} = 5x - 12z + 12 - \lambda = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = 170 - 3x - z = 0 \qquad (3)$$

Solve FOCs simultaneously for x^*, z^* and λ^*
From (2) $\lambda = 5x - 12z + 12$ plug into (1) to get the following:

$$-10 - 21x + 41(170 - 3x) = 0 \Rightarrow -10 - 21x + 6970 - 123x = 0 \Rightarrow 144x = 6960 \Rightarrow$$

$$x^* = \frac{145}{3} = 48\frac{1}{3}$$

$$\Rightarrow z^* = 170 - 3\left(\frac{145}{3}\right) = 25 \qquad \boxed{x^* = 48\frac{1}{3}; z^* = 25}$$

$\lambda^* = 5\left(\frac{145}{3}\right) - 12 \cdot 25 + 12 = \boxed{\lambda^* = -46\frac{1}{3}}$ and it implies that if a constant of the constraint increases by one unit, the value of the objective function y would approximately <u>decrease</u> by $46\frac{1}{3}$