Math 308 H & I Assignment 4

Due 14th Feb, 2009

1 Practice

Problems in this section will not be collected for grading.

1. Prove that if P is invertible, then $\text{rank}(P^{-1}AP) = \text{rank}(A)$.

2. A is a m by n matrix and $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ are some vectors. Suppose you know that $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is a basis for $\text{Im}\, A$, and $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is a basis for $\ker A$.

   (a) Find $\text{rank}\, A$.

   (b) Find n.

   (c) Find the minimum value of m.

   (d) Find the number of solutions to the matrix equation $A\mathbf{x} = \mathbf{u_2}$.

3. Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be defined as $T(a_1, a_2, a_3) = (a_1 + a_2 + a_3, 2a_1 + a_3)$. Find the matrix representation with respect to the bases: $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}, \{\mathbf{e_2}, \mathbf{e_1}\}$ for $\mathbf{R}^3$, and $\mathbf{R}^2$ respectively.

4. Let

$$A = \begin{pmatrix} 2 & 8 & 1 & 0 & 7 & 0 \\ -3 & -12 & 0 & 2 & 2 & 0 \\ 5 & 20 & -2 & -1 & 0 & 0 \end{pmatrix}$$

   Find a basis for $\text{Im}\, A$ and $\ker A$.

2 Problems

Problems in this section will be collected for grading.

1. Suppose $A, B \in M_n(\mathbf{C})$, and $AB = 0$.

   (a) Show that $\text{Im}(B) \subset \ker(A)$.

   (b) Show that $\text{rank}\, A + \text{rank}\, B \leq n$.

2. Let $P_n[x] = \{f(x) \in \mathbf{R}[x] \mid \deg f \leq n\}$, i.e. $P_n[x]$ is the set of polynomials with real coefficients of degree at most n. It is known that $P_n[x]$ is a vector space over $\mathbf{R}$.

   (a) Find a basis for $P_n[x]$.

   (b) What is $\dim(P_n[x])$?

   (c) Determine whether the polynomials:

$$x^2 + 1, x^2 + x, x^2 + x + 1, x^2 + 2x + 1$$

   are linearly independent in $P_2[x]$.

   (d) Define $D : P_n[x] \to P_n[x]$ by $D(f(x)) = f'(x)$. Show that D is a linear transformation.

   (e) Find $\ker D$.

(f) Find rank(D).

(g) Find the eigenvalues of D.

(h) Find tr(D).

(i) Find det(D).

(j) For $n = 2$. Find the matrix representation of D with respect to the basis you fond in part (a).

(k) Define $Int : P_n[x] \to P_{n+1}[x]$ by

$$Int(f(x)) = \int_0^x f(t)dt.$$

Show that Int is a linear transformation.

(l) Find rank(Int).

3.

(a) Show that $\text{tr} : M_n(\mathbf{R}) \to \mathbf{R}$ is a linear transformation.

(b) Find dim(ker(tr)).