The Lorenz Attractor

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Abstract

The Lorenz System, introduced by E. N. Lorenz in 1963, is a system of differential equations which offers explanation to unpredictable behavior of the weather. Lorenz investigated two-dimensional fluid cell which was heated from below and cooled from above. The fluid motion can be described by a system of differential equations involving infinitely many variables. After simplification, Lorenz was able to reduce the problem into a three-dimensional system with three parameters. The stability of the system can be analyzed by a method known as *linear stability analysis*. The Lorenz System shows that all non-equilibrium solutions tend eventually to the same complicated set, known as *Lorenz Attractor*. The r dependence of the attractor will also be discussed.

Problem

Chaos theory deals with the behavior of certain nonlinear dynamical systems that exhibit chaos under certain conditions. Such systems include the atmosphere and turbulent fluids. They share the fundamental property in which their behavior is crucially *sensitive to their initial conditions.* This phenomenon is also known as the *butterfly effect.*

Lorenz was interested in modeling Earth's atmosphere and determining how well weather could be predicted. Such weather systems in which the initial conditions cover some area in space have an infinite number of parameters, and can be described by partial differential equations.

The original problem is a 7^{th} order system whose numerical solution holds aperiodic behavior. B. Saltzman formed a system of equations for a simple type of convection. He noticed that the solution appeared to undergo irregular fluctuations, implying unsteady convection. A model of atmospheric convection, the Lorenz system was obtained based on Saltzman's equations. Lorenz observed that four of the variables approaches zero eventually and hence arrived at the system known as the *Lorenz equations*.

Simplification

Lorenz made the assumption that the fluid velocity could be described by a single roll and that the temperature in the layer could be described by a steady state solution and two time dependent modes.

$x(t), y(t)$ and $z(t)$ denotes the rotational speed of the convection cell, the temperature difference between the ascending and descending currents, and the distortion of the vertical temperature from the linear conduction profile from linearity (i.e. the instability of the air), respectively. If x is positive, the cell spins clockwise. If x is negative, it spins the other direction. The bigger the absolute value of x, the faster it spins. Furthermore, if y is positive the left side of the convection cell is hot, the right side cold (and vice versa). Finally, z is always positive. Higher values imply that the top of the convection cell is hotter. If both y and z are zero then the temperature is just layered with hot material at the bottom and cold material on top. Also, there is no convection when z is zero. We only need to solve for the values of x, y, z through time.

The Lorenz equations arise from substituting into more general equations of convection and arriving at a dynamical system for the three time dependent variables.

Mathematical Model

The problem can be translated into the Lorenz equations below:

$$\frac{dx}{dt} = \sigma(-x + y)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = -bz + xy$$

The property of the fluid is represented by σ, the Prandtl number, while b denotes the geometric factor. The Rayleigh number r is just a measure of temperature difference between the lower and upper plate. If $r < 1$, the layer is being heated too gently to convect. For $r = 1.1$ the convection cell convects at a steady velocity clockwise.

As we increase the Rayleigh number to near 25, all initial conditions will eventually settle down to one of two steady states, a cell that either rotates clockwise of counter clockwise. As the Rayleigh number is increased, it takes a relatively longer time to settle down, causing the final speed of the cell to be higher. At a critical Rayleigh number (24.74), the roll becomes unstable and the system enters a chaotic cycle where the roll continuously oscillates in time and flips directions in a chaotic yet predictable manner. For this problem, it is normalized so that $r = 1$ corresponds to the critical value for the first occurrence of convection.

Solution

The solution to Lorenz equation can be analysed through the investigation of the equilibrium points of the system. The equilibrium points are $P_1 = (0,0,0)$, $P_2 = (\sqrt{b(r-1)}, \sqrt{b(r-1}, r-1)$ and $P_3 = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$. Next, the stability of these equilibrium points is analyzed. The Jacobian matrix A for the system is:

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z* & -1 & -x* \\ y* & x* & -b \end{pmatrix}$$

Nontrivial solutions are possible only if $\det(A - \lambda I) = 0$. In other words:

$$\det \begin{pmatrix} -\sigma - \lambda & \sigma & 0 \\ r - z* & -1 - \lambda & -x* \\ y* & x* & -b - \lambda \end{pmatrix} = 0$$

Therefore, the eigenvalues for $P_1 = (x*, y*, z*) = (0,0,0)$ are:

$$\lambda_1 = -b$$
$$\lambda_2 = \frac{1}{2}[-(\sigma+1) + \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}]$$
$$\lambda_3 = \frac{1}{2}[-(\sigma+1) - \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}]$$

P_2 and P_3 gives the equation $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$, in which gives us the solutions:

$$a_0 = 2\sigma b(r-1)$$
$$a_1 = b(r+\sigma)$$
$$a_2 = \sigma + b + 1$$

Assuming that there exists a critical value r_c for r in which the following holds true:

$$\text{Re}\,\lambda < 0 \text{ for } 1 < r < r_c$$

$$\text{Re}\,\lambda > 0 \text{ for } r > r_c$$

Hence, this will lead to the solution for r_c:

$$r_c = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right)$$

Lorenz used the following set of parameter values:

$$\sigma = 10,\ \ b = 8/3$$

which yields $r_c = 470/19 \cong 24.737$. The solved Lorenz equations can now be written as:

$$\frac{dx}{dt} = -10x + 10y$$
$$\frac{dy}{dt} = 28x - y - xz$$
$$\frac{dz}{dt} = -\frac{8}{3}z + xy$$

Results

Using the MATLAB program in the Appendix to plot the Lorenz system, we get the following plot for the initial conditions $(x, y, z) = (0,1,0)$ and $(0,-1,0)$.

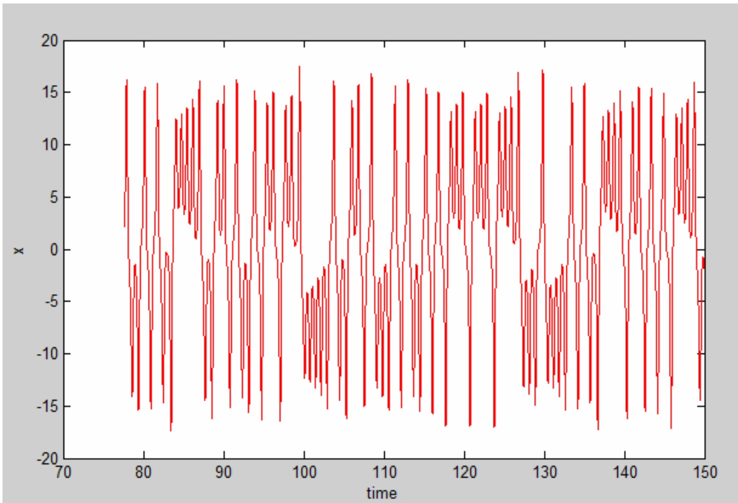

Fig. 1
Plot for time series when r=28

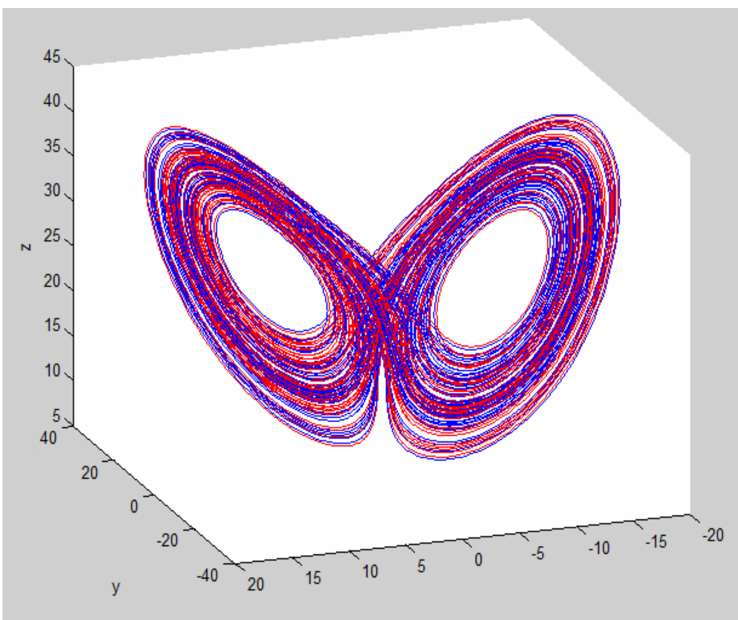

Fig. 2
Plot for phase space when r=28

The phase space plot for the Lorenz equations depicts the well-known Lorenz butterfly shape. The trajectories are attracted to near the equilibrium points P_2 and P_3. A trajectory may wind around P_2 several times before being repelled to P_3, and winding around it, and so on.

For $0 < r < 1$, the origin $(0,0,0)$ is the only equilibrium point and all points are attracted to it. This means that $(x^*, y^*, z^*) = (0,0,0)$ is nonlinearly stable for $r < 1$. At $r = 1$, a pitchfork bifurcation occurs. The origin becomes unstable and two stable equilibrium points appear. For $1 < r < 13.925$, the unstable manifold of the origin connects to the equilibrium points. Fig. 3 shows $r = 10$.

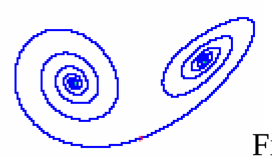

Fig. 3

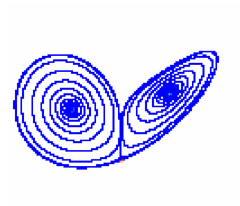

Fig. 4

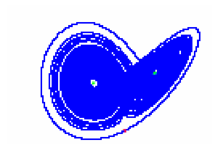

Fig. 5

In the case of $r = r_0 = 13.926$, the unstable manifold becomes double asymptotic to the origin, as shown by Fig. 4. At the parameter $r = 28$, the Lorenz attractor is observed (Fig. 5). For large parameters of r, the attractor can be single periodic orbit. The plots below show the behavior for r=100.

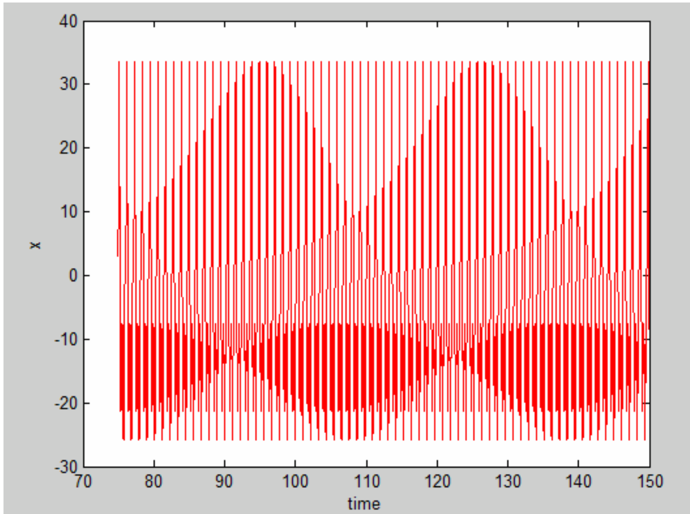

Fig. 6
Plot for time series when r=100

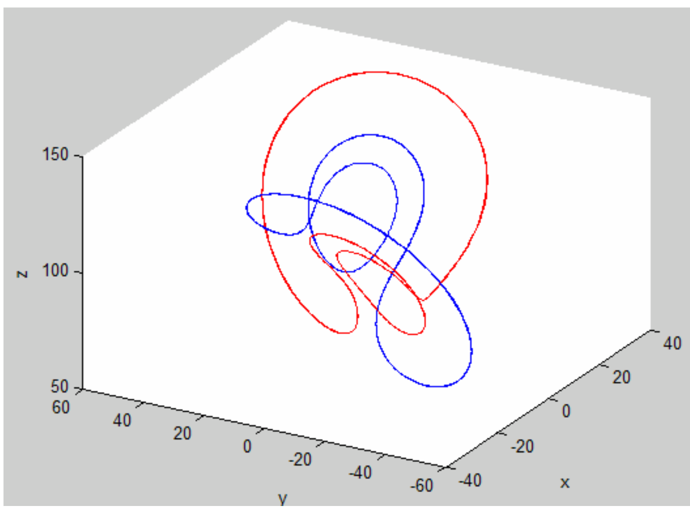

Fig. 7
Plot for phase space when r=100

Improvement

Much attention has been placed on sensitivity to small changes in initial condition as a limit to predictability. Techniques to counter such problems include ensemble weather prediction, where an ensemble of initial conditions, within some tolerance of the analysis, is run forward under the model to get an impression of the likely range of future states. While model error has been studied in the context of weather forecasting, a satisfactory method for measuring its effect on forecast accuracy has not been found (Houtekamer et al., 1996). A lower bound can be obtained by comparing several models. However the models share common assumptions about the physics, and an upper bound is not known.

A critical question is whether initial conditions exist where one type of error effectively offsets the other, such that the trajectory *shadows* (stays within a specified radius of *r*). The existence of trajectories that show such characteristic within the observational tolerance is a useful measure of model quality.

By investigating the relationship between model error and shadow times by linearising the dynamics for a shadow trajectory around a target orbit, we can arrive at the derivation of a fundamental shadowing law which gives a lower bound on the radius at which a model can be expected to shadow. The methods can then be applied to a medium-dimensional system model known as the *constant model*.

The equation for the Lorenz one-level system used as the *constant model* is:

$$\frac{dx_i}{dt} = x_{i-1}(x_{i+1} - x_{i-2}) - x_i + F$$

The index *i* is cyclic so that $x_{i-8} = x_{i+8} = x_i$. This 8-dimensional model consists of 8 variables x_i which is an atmospheric quantity like temperature distributed around a circle. It simulates advection, internal damping and a constant external forcing.

The equations for the two-level system used as the target system are:

$$\frac{d\tilde{x}_i}{dt} = \tilde{x}_{i-1}(\tilde{x}_{i+1} - \tilde{x}_{i-2}) - \tilde{x}_i + F - \frac{hc}{b}\sum_{j=1}^{m} \tilde{y}_{i,j}$$

$$\frac{d\tilde{y}_{i,j}}{dt} = cb\tilde{y}_{i,j+1}(\tilde{y}_{i,j-1} - \tilde{y}_{i,j+2}) - c\tilde{y}_{i,j} + \frac{hc}{b}\tilde{x}_i$$

for $i = 1, ..., n$ and $j = 1, ..., m$. The variables are cyclic so that $\tilde{y}_{i+n,j} = \tilde{y}_{i,j}$ and $\tilde{y}_{i,j-m} = \tilde{y}_{i-1,j}$. The coefficients are $b = c = F = 10$ for which the $\tilde{y}$'s tend to fluctuate ten times more rapidly but with ten times smaller magnitude than the $\tilde{x}$'s. This improved model is a variant of the *constant model*, where the forcing, instead of being constant, also employs a term linear in x so as to minimize the expected tendency error over the attractor. It shadows substantially longer than the original model. The situation is analogous to that encountered in real weather models, where a parameterization (the constant forcing) is adopted to model connective-scale fluctuations (the fine-scale variables).

Furthermore, by measuring the contribution of model error to forecast error, we arrive at certain simple explanations for several observed phenomena. These include the negative curvature, square root growth of error, which later combines with displacement error to form a nearly linear growth rate; the inability of models to shadow within a certain radius. Drift (*magnitude of integrated tendency error evaluated over a segment of a target orbit)* and shadowing experiments may further reveal the impact of model error on operational forecasts, suggest methods to reduce it, and provide a realistic assessment of model quality.

Conclusions

The result of the Lorenz equations produce *deterministic chaos* since it is non-random and we know everything about how it will change instantaneously. It becomes *chaotic* for high enough Rayleigh numbers, with small changes in the initial conditions leading to very different behavior eventually due to small differences grow in a nonlinear feedback with time.

Lorenz intended only to derive a simplified set of equations based on more complicated systems which display aperiodicity and extreme sensitivity to initial conditions. His equations are not a good model since it is based on a set of two partial differential equations describing the fluid convection in two dimensions. Furthermore, when r becomes large, the terms which are omitted by Lorenz are too critical to be ignored.

While the effects of chaos eventually lead to loss of predictability, it happens only over long time scales. Model quality can be done through the use of local model drift, which is a good measure of model error. It describes how model error evolves over time and affects predictability, hence offering better insight on a more accurate weather predictability models.

```
function doall=lorenz
clear all; close all;
% Solve Lorenz system using ode45 (Runga-Kutta package of matlab)

%  dx/dt = sigma*(y-x)
%  dy/dt = r*x-y-x*z
%  dz/dt = x*y-b*z

sigma=10.0;
b=8.0/3.0;
r=28;

delta_t=0.01;

% integration time:
T=150.0;
%set error bounds for integration
options=odeset('RelTol',10^(-6),'AbsTol',[10^(-6)*ones(1,3)]);

% integrate:
% initial conditions (x,y,z)=[0.00001 0 0]
 [t,x] = ode45(@lor,[0 T],[0.00001 0 0],options,[sigma; b ;r]);
 % initial conditions (x,y,z)=[-0.00001 0 0]
 [t,xl] = ode45(@lor,[0 T],[-0.00001 0 0],options,[sigma; b ;r]);

% plot simple time series:
b=length(t(:))/2;
e=length(t(:));
plot(t(b:e),x(b:e,1))
plot(t(b:e),xl(b:e,1),'r')
xlabel('time')
ylabel('x')

% plot phase space:
figure
plot3(x(b:e,1),x(b:e,2),x(b:e,3))
hold
plot3(xl(b:e,1),xl(b:e,2),xl(b:e,3),'r')
xlabel('x')
ylabel('y')
zlabel('z')

function dy = lor(t,x,kh)
sigma=kh(1); b=kh(2); r=kh(3);
dy=[sigma*(x(2)-x(1)),r*x(1)-x(2)-x(1)*x(3),x(1)*x(2)-b*x(3)]';
```

References

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