

Lag Operator

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▪ How does the **lag operator** L work? It is defined this way: $y_{t-1} = Ly_t$, i.e. you lag the variable by one period by multiplying it with the lag operator L . More examples:

▪ 1st difference: $(1-L)y_t = y_t - y_{t-1}$

▪ Lagging by p periods: $L^p y_t = y_{t-p}$

▪ Lagging a constant: $La = a$

▪ 1st difference of 1st difference: $(1-L)^2 y_t = (1+L^2-2L)y_t = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})$

▪ Now we can define a **distributed lag model**: $y_t = \alpha + \sum_{i=0}^{\infty} \beta_i L^i x_t + u_t = \alpha + B(L)x_t + u_t$,

where $B(L) = \beta_0 + \beta_1 L + \beta_2 L^2 + \dots$ is a **lag operator polynomial**.

▪ A special case of a lag operator polynomial is $A(L) = 1 + aL + (aL)^2 + \dots = \sum_{i=0}^{\infty} (aL)^i$.

Provided that $|a| < 1$, we can think of the above as a geometric sum: $A(L) = \frac{1}{1-aL}$.

▪ Why is it true? Consider this

$$\frac{1-aL}{1-aL} = (1+aL+(aL)^2+\dots)(1-aL)$$

$$= (1+aL+(aL)^2+\dots) - (aL+(aL)^2+\dots) = 1$$

▪ If our distributed lag model has the form: $y_t = \alpha + \beta \sum_{i=0}^{\infty} \gamma_i L^i x_t + u_t$, we can rewrite it

as $y_t = \alpha + \beta(1-\gamma L)^{-1} x_t + u_t$. That is, we write the model as a **moving-average** model.

▪ We can multiply both sides of the model by $(1-\gamma L)$:

$$(1-\gamma L)y_t = (1-\gamma L)\alpha + \beta x_t + (1-\gamma L)u_t \text{ or } y_t = (1-\gamma)\alpha + \beta x_t + \gamma y_{t-1} + (1-\gamma L)u_t.$$

Example 1¹

▪ GNP: $y_t = \alpha(L)\varepsilon_t + \beta(L)\eta_t$, where $\alpha(L) = \sum_{j=0}^{\infty} \alpha_j L^j$ and $\beta(L) = \sum_{j=0}^{\infty} \beta_j L^j$

▪ Policy: $m_t = \gamma(L)\varepsilon_t$ where $\gamma(L) = \sum_{j=0}^{\infty} \gamma_j L^j$.

▪ η_t and ε_t are mutually uncorrelated and serial uncorrelated random variables with mean zeros and variances σ_η^2 and σ_ε^2 .

¹ This example is a modified version of “**The Observational Equivalence of Natural and Unnatural Rate Theories of Macroeconomics**” by Thomas J. Sargent (*Journal of Political Economy*, Vol. 84, No. 3. (Jun., 1976), pp. 631-640.)

- Manipulate the policy equation: $\gamma^{-1}(L)m_t = \varepsilon_t$, and plug it in the GNP equation $y_t = \alpha(L)\gamma^{-1}(L)m_t + \beta(L)\eta_t = h(L)m_t + \beta(L)\eta_t$.
 - Further manipulate the equation we have $\beta^{-1}(L)y_t = \beta^{-1}(L)h(L)m_t + \eta_t$, which can be rewritten as $y_t = \sum_{i=0}^{\infty} a_i m_{t-i} + \sum_{i=0}^{\infty} b_i y_{t-i} + \eta_t = a(L)m_t + b(L)y_{t-1} + \eta_t$ (**Result 1**), where $a(L) = \sum_{i=0}^{\infty} a_i L^i = \beta^{-1}(L)h(L)$ and $1 - Lb(L) = 1 - \sum_{i=0}^{\infty} b_i L^{i+1} = \beta^{-1}(L)$
 - Hey, provided that Result 1 remains unchanged under policy change, what if we set our policy as $a(L)m_t = -b(L)y_t$ (a **feedback rule**) to eliminate most of the variance in y_t ? Wouldn't it be great?
 - Don't be too happy. We can rewrite the policy as $m_t = \gamma_0 \varepsilon_t + E_{t-1}m_t$, or $\varepsilon_t = \frac{1}{\gamma_0}(m_t - E_{t-1}m_t)$. Putting it into the GNP equation, we have $y_t = \alpha(L)\frac{1}{\gamma_0}(m_t - E_{t-1}m_t) + \beta(L)\eta_t$ or $y_t = c(L)(m_t - E_{t-1}m_t) + b(L)y_{t-1} + \eta_t$ (**Result 2**), where $c(L) = \beta^{-1}(L)\alpha(L)\frac{1}{\gamma_0}$.
 - Notice that Result 2 has the same error term and same RHS, and it should fit as good as Result 1.
 - If we assume it is Result 2 that remains unchanged as the policy is altered, then any policy rule that uses past information is "useless". For example, we have a rule of the form $m_t = p_1 m_{t-1} + p_2 m_{t-2} + \dots + q_1 y_{t-1} + q_2 y_{t-2} + \dots$, then $m_t = E_{t-1}m_t$, and the policy has no impact on y_t at all.
 - So the moral from this exercise: if you assume it is Result 1 that stays the same under policy change, then policy rocks (i.e. policy is non-neutral); if you assume it is Result 2 that stays the same under policy change, then policy sucks (i.e. policy is neutral).
 - Or, by estimating equations like Result 1 and Result 2 cannot tell you whether policy is good.
- Example 2**
- Let's consider the ARMA(1,1) model: $y_t = \phi y_{t-1} + u_t - \theta u_{t-1}$. Rewriting it with the lag operator: $(1 - \phi L)y_t = (1 - \theta L)u_t$.
 - What will happen if $\phi \approx \theta$ (the model is not well identified)? Canceling out?
 - I do a Monte Carlo with 500 runs for the DGP $y_t = ky_{t-1} + u_t$ of sample size 1000, and fit an ARMA(1,1) model in each run. I do a t -test for $\theta = 0$ at 5% level. If things are good, I should reject the true null about 5% of the time. Here is what will happen:
 When $k=0.5$, rejection rate=4.4%
 When $k=0.1$, rejection rate=18.8%
 When $k=0.05$, rejection rate=29.4%
 When $k=0.01$, rejection rate=39.6%