Topic 12 – Simultaneous-Equations Models

- Remember the demand-supply example I gave two weeks ago? That is what we call a simultaneous-equations model.

- To understand how these models work, it is important to understand the meaning of exogeneity.

- Engle, Hendry, and Richard (1983)[9] propose three different levels of exogeneity: weak, strong, and super exogeneity. The paper is technically difficult, but it will be useful to have at least heard of the jargons.

- Consider the joint density of two sets of observed variables: $\mathbf{x}_t = (\mathbf{y}_t' \quad \mathbf{z}_t')'$. Denote the past values of $\mathbf{x}_t$ as $\mathbf{X}_{t-1}$, and the parameters of the joint density function as $\boldsymbol{\lambda}$. We can always write the joint density as

$$D(\mathbf{x}_t \mid \mathbf{X}_{t-1}, \boldsymbol{\lambda}) = D(\mathbf{y}_t, \mathbf{z}_t \mid \mathbf{X}_{t-1}, \boldsymbol{\lambda}) = D(\mathbf{y}_t \mid \mathbf{z}_t, \mathbf{X}_{t-1}, \boldsymbol{\lambda}_1) D(\mathbf{z}_t \mid \mathbf{X}_{t-1}, \boldsymbol{\lambda}_2).$$

- If 1) $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are not subject to cross-restrictions (there is a “sequential cut”), and 2) the parameters of interest $\boldsymbol{\psi}$ can be determined from the conditional density alone (i.e. $\boldsymbol{\psi} = f(\boldsymbol{\lambda}_1)$), then we say $\mathbf{z}_t$ is **weakly exogenous** for $\boldsymbol{\psi}$.

- If in addition to 1) and 2), $\mathbf{y}_t$ also does not Granger cause $\mathbf{z}_t$ (loosely speaking, it means given past values of $\mathbf{z}_t$, the past values of $\mathbf{y}_t$ do not explain the current $\mathbf{z}_t$), then we say $\mathbf{z}_t$ is **strongly exogenous** for $\boldsymbol{\psi}$.

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[9] “Exogeneity” by **Robert F. Engle; David F. Hendry; Jean-Francois Richard** (*Econometrica*, Vol. 51, No. 2. (Mar., 1983), pp. 277-304.)

▪ If in addition to 1) and 2), $\boldsymbol{\lambda}_1$ is invariant to changes in $\boldsymbol{\lambda}_2$, or the conditional distribution is invariant to any change in the marginal distribution, then we say $\mathbf{z}_t$ is **super exogenous** for $\boldsymbol{\psi}$.

▪ Let's have fun with some horrendous notations. We have an M equations and M endogenous variables model in structural form as

$$\gamma_{11}y_{t1}+\cdots+\gamma_{M1}y_{tM}+\beta_{11}x_{t1}+\cdots+\beta_{K1}x_{tK}=\varepsilon_{t1}$$

$$\gamma_{12}y_{t1}+\cdots+\gamma_{M2}y_{tM}+\beta_{12}x_{t1}+\cdots+\beta_{K2}x_{tK}=\varepsilon_{t2}$$

…

$$\gamma_{1M}y_{t1}+\cdots+\gamma_{MM}y_{tM}+\beta_{1M}x_{t1}+\cdots+\beta_{KM}x_{tK}=\varepsilon_{tM}$$

▪ The endogenous variables are $y_{t1},...,y_{tM}$, the exogenous variables are $x_{t1},...,x_{tK}$ (which usually include a constant and some predetermined/lagged variables), and the structural disturbances are $\varepsilon_{t1},...,\varepsilon_{tM}$.

▪ In matrix notations

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_M \end{bmatrix}_t \begin{bmatrix} \gamma_{11} & \gamma_{12} & \vdots & \gamma_{1M} \\ \gamma_{21} & \gamma_{22} & \vdots & \gamma_{2M} \\ & & \vdots & \\ \gamma_{M1} & \gamma_{M2} & \cdots & \gamma_{MM} \end{bmatrix} + \begin{bmatrix} x_1 & x_2 & \cdots & x_K \end{bmatrix}_t \begin{bmatrix} \beta_{11} & \beta_{12} & \vdots & \beta_{1M} \\ \beta_{21} & \beta_{22} & \vdots & \beta_{2M} \\ & & \vdots & \\ \beta_{K1} & \beta_{K2} & \cdots & \beta_{KM} \end{bmatrix}$$
$$= \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_M \end{bmatrix}_t$$

Or more clearly, $\mathbf{y}_t'\boldsymbol{\Gamma}+\mathbf{x}_t'\mathbf{B}=\boldsymbol{\varepsilon}_t'$.

- We can solve for the reduced form of the above system:

$$\mathbf{y}_t' = -\mathbf{x}_t'\mathbf{B}\boldsymbol{\Gamma}^{-1} + \boldsymbol{\varepsilon}_t'\boldsymbol{\Gamma}^{-1} = \mathbf{x}_t'\boldsymbol{\Pi} + \mathbf{v}_t'$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_M \end{bmatrix}_t \begin{bmatrix} \pi_{11} & \pi_{12} & \vdots & \pi_{1M} \\ \pi_{21} & \pi_{22} & \vdots & \pi_{2M} \\ & & \vdots & \\ \pi_{K1} & \pi_{K2} & \cdots & \pi_{KM} \end{bmatrix} + \begin{bmatrix} v_1 & v_2 & \cdots & v_M \end{bmatrix}_t$$

- Obviously, for the above manipulation to work, we need $\boldsymbol{\Gamma}$ to be non-singular (completeness condition).

- We have some familiar assumptions for the structural disturbances: $E\left(\boldsymbol{\varepsilon}_t \mid \mathbf{x}_t\right) = \mathbf{0}$, $E\left(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t' \mid \mathbf{x}_t\right) = \boldsymbol{\Sigma}$, and $E\left(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_s' \mid \mathbf{x}_t, \mathbf{x}_s\right) = \mathbf{0}$ for all $t \neq s$.

- Due to the above assumptions, we know the following about the reduced-form disturbances: $E\left(\mathbf{v}_t \mid \mathbf{x}_t\right) = \boldsymbol{\Gamma}^{-1}{}'\mathbf{0} = \mathbf{0}$, $E\left(\mathbf{v}_t\mathbf{v}_t' \mid \mathbf{x}_t\right) = \boldsymbol{\Gamma}^{-1}{}'\boldsymbol{\Sigma}\boldsymbol{\Gamma}^{-1} = \boldsymbol{\Omega}$.

- For the reduced-form equations, we can estimate the parameters consistently by OLS and get $\hat{\boldsymbol{\Pi}}$ and $\hat{\boldsymbol{\Omega}}$. Why?

- The problem here is how to go from $\hat{\boldsymbol{\Pi}}$ and $\hat{\boldsymbol{\Omega}}$ and learn something about the structure $\left[\mathbf{B}, \boldsymbol{\Gamma}, \boldsymbol{\Sigma}\right]$.

- Or, to put it more technically, the problem is to use estimates from the reduced-form equations to estimate the parameters in the structural equations.

- The model we have now is too general for us to do that: the structure $\left[\mathbf{B}, \boldsymbol{\Gamma}, \boldsymbol{\Sigma}\right]$ and the structure $\left[\mathbf{BF}, \boldsymbol{\Gamma}\mathbf{F}, \mathbf{F}'\boldsymbol{\Sigma}\mathbf{F}\right]$ are observationally equivalent, where $\mathbf{F}$ is any conformable non-singular matrix.

- More specifically, we have $M^2 + KM + \frac{1}{2}M\left(M+1\right)$ parameters to estimate in the structural model, while we only have $KM + \frac{1}{2}M\left(M+1\right)$ parameters available in the reduced-form model.
- We need more assumptions to get us out of the problem.
- **Normalizations**: It is harmless for each equation to have a coefficient of 1, and that will save us M structural parameters.
- **Identities**: For some equations we know the parameters exactly and no error term is needed (e.g. quantity demanded=quantity supplied).
- **Exclusions**: In both $\mathbf{B}$ and $\mathbf{\Gamma}$ we can insert some zeros, meaning that we exclude some variables from some equations (e.g. income does not affect quantity supplied).
- **Linear restrictions**: We can impose some restrictions among the parameters in $\mathbf{B}$ and $\mathbf{\Gamma}$ (e.g. the income elasticities for apple and orange sum up to one).
- **Disturbance covariance matrix restrictions**: We may assume some shocks are uncorrelated (supply shock is not correlated with demand shock).
- Now look at the *j*th equation: $\mathbf{y}_t{}'\mathbf{\Gamma}_j + \mathbf{x}_t{}'\mathbf{B}_j = \varepsilon_{jt}$, or written more explicitly in terms of the *j*th endogenous variable: $y_{jt} = \mathbf{Y}_{jt}{}'\boldsymbol{\gamma}_j + \mathbf{Y}_{jt}^*{}'\boldsymbol{\gamma}_j^* + \mathbf{x}_{jt}{}'\boldsymbol{\beta}_j + \mathbf{x}_{jt}^*{}'\boldsymbol{\beta}_j^* + \varepsilon_{jt}$. The variables with $*$ are variables that are excluded from the equation (i.e. having zero coefficients).
- We have a vector of M_j included endogenous variables $\mathbf{Y}_{jt}$, a vector of M_j^* excluded endogenous variables $\mathbf{Y}_{jt}^*$, a vector of K_j included exogenous variables $\mathbf{x}_{jt}$, and a vector of K_j^* excluded exogenous variables $\mathbf{x}_{jt}^*$. Notice that $M_j + M_j^* + 1 = M$ and $K_j + K_j^* = K$.

- So we have $\mathbf{\Gamma}_j = \begin{bmatrix} 1 \\ -\boldsymbol{\gamma}_j \\ \mathbf{0} \end{bmatrix}$ and $\mathbf{B}_j = \begin{bmatrix} -\boldsymbol{\beta}_j \\ \mathbf{0} \end{bmatrix}$.

- Now go back to the reduced-form, and we partition the variables similarly:

$$\begin{bmatrix} y_{jt} & \mathbf{Y}_{jt}' & \mathbf{Y}_{jt}^{*}' \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{jt}' & \mathbf{x}_{jt}^{*}' \end{bmatrix} \begin{bmatrix} \boldsymbol{\pi}_j & \mathbf{\Pi}_j & \bar{\mathbf{\Pi}}_j \\ \boldsymbol{\pi}_j^{*} & \mathbf{\Pi}_j^{*} & \bar{\mathbf{\Pi}}_j^{*} \end{bmatrix} + \begin{bmatrix} v_{jt} & \mathbf{V}_{jt} & \mathbf{V}_{jt}^{*} \end{bmatrix}.$$

- Remember what we have $\mathbf{\Pi} = -\mathbf{B}\mathbf{\Gamma}^{-1} \to \mathbf{\Pi}\mathbf{\Gamma} = -\mathbf{B}$. We can pick out the *j*th equation:

$$\mathbf{\Pi}\mathbf{\Gamma}_j = -\mathbf{B}_j \to \begin{bmatrix} \boldsymbol{\pi}_j & \mathbf{\Pi}_j & \bar{\mathbf{\Pi}}_j \\ \boldsymbol{\pi}_j^{*} & \mathbf{\Pi}_j^{*} & \bar{\mathbf{\Pi}}_j^{*} \end{bmatrix} \begin{bmatrix} 1 \\ -\boldsymbol{\gamma}_j \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\beta}_j \\ \mathbf{0} \end{bmatrix}.$$ We can finally get two sets of equations:

$\boldsymbol{\pi}_j - \mathbf{\Pi}_j \boldsymbol{\gamma}_j = \boldsymbol{\beta}_j$ (K_j equations) and $\boldsymbol{\pi}_j^{*} - \mathbf{\Pi}_j^{*} \boldsymbol{\gamma}_j = \mathbf{0}$ (K_j^{*} equations).

- The second sets of equations have K_j^{*} equations and M_j unknowns, and so we need $K_j^{*} \geq M_j$ to have solutions (**order condition**). In words, the number of excluded exogenous variables should be at least as large as the number of included endogenous variables.