ECON582 Econometrics III – Midterm Exam

26th April 2007, 1:30 pm-2:50 pm

It is a closed-book exam, and you do not need any statistical table

Keep your answer short

Total points=70

Short Questions (Answer all questions)

Question 1 (Ordinary Least Squares) [10 points] – Consider the regression model with one RHS variable $y_i = \beta x_i + \varepsilon_i$, where all the classical linear regression assumptions hold. What is the probability limit of the OLS estimator $\hat{\beta}$? (5 points) What is the asymptotic distribution of the OLS estimator $\hat{\beta}$? (5 points) Show your steps clearly, and you can assume that some relevant law of large number (LLN) and central limit theorem (CLT) hold.

Answer: Refer to Greene Chapter 5

Grading Policy: You get full credits as long as you show the main steps of the derivation.

Question 2 (Ordinary Least Squares) [10 points] – Consider the regression model with one RHS variable $y_i = \beta x_i + \varepsilon_i$. Prove that the OLS residual $\hat{\varepsilon}_i = y_i - \hat{\beta} x_i$ is orthogonal to the RHS variable x_i, i.e. $\sum_{i=1}^{n} x_i \hat{\varepsilon}_i = 0$. Does this result imply the OLS assumption $\sum_{i=1}^{n} x_i \varepsilon_i = 0$, where ε_i is the error term? Explain.

Answer: $\sum_{i=1}^{n} x_i \hat{\varepsilon}_i = \sum_{i=1}^{n} x_i \left(y_i - \hat{\beta} x_i \right) = \sum_{i=1}^{n} x_i y_i - \hat{\beta} \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i - \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} \sum_{i=1}^{n} x_i^2 = 0$

The orthogonality between the RHS and the residual defines the OLS estimator, and the orthogonality between the RHS and the error is an assumption, which we cannot test because the error is unobservable.

Grading Policy: The above proof does not require any expectation and assumption. All you need is the definition of the OLS estimator.

Question 3 (Instrumental Variable Estimation) [10 points] – Consider the regression model with one RHS variable $y_i = \beta x_i + \varepsilon_i$, where all the classical linear regression assumptions hold. Now x_i is not correlated with ε_i, but you use the IV estimator $\hat{\beta}^{IV}$ anyway. Which is bigger, $\text{var}\left(\hat{\beta}^{OLS}\right)$ or $\text{var}\left(\hat{\beta}^{IV}\right)$? Explain briefly.

Answer: Due to the Gauss-Markov Theorem (we can use it as all the classical assumptions are met), $\text{var}\left(\hat{\beta}^{IV}\right)$ must be larger.

Grading Policy: Writing down the proof on page 80 in Greene is acceptable, though it is not technically correct: the proof is for the asymptotic variances only. The Gauss-Markov Theorem is more general.

Question 4 (Instrumental Variable Estimation) [10 points] – Consider the regression model with one RHS variable $y_i = \beta x_i + \varepsilon_i$, where x_i IS correlated with ε_i, and all the other classical linear regression assumptions hold. Is it necessarily better to use IV (assuming that the instrument is <u>valid</u>) in small sample? (5 points) What about in large sample? (5 points) Explain your answers briefly.

Answer: <u>Small sample and weak instrument</u>: You just have to briefly mention the main points of the Nelson-Startz paper.
<u>Large sample and weak instrument</u>: The problems in the Nelson-Startz paper go away with a large sample, but as Mean-squared error = Bias^2 + Var^2, we still face a tradeoff between bias (the OLS is worse) and variance (the IV is worse).

Grading Policy: You don't need to write too much as long as you get the main points.

Long Questions (Choose 2 out of 3)

Question 5 (Measurement Errors) [15 points] – Consider the true model $y_i^* = \beta x_i^* + \varepsilon_i$, where all the classical linear regression assumptions hold. All variables have zero mean. Unluckily, observations on the true variables may not be available. Find the probability limit of the OLS estimator $\hat{\beta}$ for the following situations:

a) The LHS variable is measured with error $y_i = y_i^* + v_i$, $E\left(v_i \mid y_i^*, x_i^*\right) = 0$, $E\left(v_i^2 \mid y_i^*, x_i^*\right) = \sigma_v^2$ (5 points).

b) The RHS variable is measured with error $x_i = x_i^* + u_i$, $E\left(u_i \mid y_i^*, x_i^*\right) = 0$, $E\left(u_i^2 \mid y_i^*, x_i^*\right) = \sigma_u^2$ (5 points).

c) Both the LHS and RHS variables are measured with error $y_i = y_i^* + v_i$, $x_i = x_i^* + u_i$, $E\left(v_i \mid y_i^*, x_i^*\right) = E\left(u_i \mid y_i^*, x_i^*\right) = E\left(u_i v_i \mid y_i^*, x_i^*\right) = 0$, $E\left(v_i^2 \mid y_i^*, x_i^*\right) = \sigma_v^2$, $E\left(u_i^2 \mid y_i^*, x_i^*\right) = \sigma_u^2$ (5 points).

Answer

a) Solve for the OLS estimator $\hat{\beta} = \dfrac{\sum_{i=1}^n x_i^* y_i}{\sum_{i=1}^n x_i^{*2}} = \dfrac{\sum_{i=1}^n x_i^* \left(\beta x_i^* + \varepsilon_i + v_i\right)}{\sum_{i=1}^n x_i^{*2}} = \beta + \dfrac{\sum_{i=1}^n x_i^* \varepsilon_i + \sum_{i=1}^n x_i^* v_i}{\sum_{i=1}^n x_i^{*2}}$

Taking probability limit: $p\lim \hat{\beta} = \beta + p\lim \dfrac{\sum_{i=1}^n x_i^* \varepsilon_i / n + \sum_{i=1}^n x_i^* v_i / n}{\sum_{i=1}^n x_i^{*2} / n} = \beta$.

b) Solve for the OLS estimator

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i y_i^*}{\sum_{i=1}^n x_i^2} = \frac{\sum_{i=1}^n \left(x_i^* + u_i\right)\left(\beta x_i^* + \varepsilon_i\right)}{\sum_{i=1}^n \left(x_i^* + u_i\right)^2} = \frac{\beta \sum_{i=1}^n x_i^{*2} + \sum_{i=1}^n x_i^* \varepsilon_i + \beta \sum_{i=1}^n x_i^* u_i + \sum_{i=1}^n \varepsilon_i u_i}{\sum_{i=1}^n \left(x_i^* + u_i\right)^2}$$

Taking probability limit

$$p\lim\hat{\beta}=\frac{\beta\sum_{i=1}^{n}x_i^{*2}/n+\sum_{i=1}^{n}x_i^{*}\varepsilon_i/n+\beta\sum_{i=1}^{n}x_i^{*}u_i/n+\sum_{i=1}^{n}\varepsilon_i u_i/n}{\sum_{i=1}^{n}\left(x_i^{*}+u_i\right)^2/n}=\frac{\beta\sigma_{x^*}^2}{\sigma_{x^*}^2+\sigma_u^2}$$

c) Solve for the OLS estimator

$$\hat{\beta}=\frac{\sum_{i=1}^{n}x_i y_i}{\sum_{i=1}^{n}x_i^2}=\frac{\sum_{i=1}^{n}\left(x_i^{*}+u_i\right)\left(\beta x_i^{*}+\varepsilon_i+v_i\right)}{\sum_{i=1}^{n}\left(x_i^{*}+u_i\right)^2}=\frac{\beta\sum_{i=1}^{n}x_i^{*2}+\sum_{i=1}^{n}x_i^{*}\varepsilon_i+\beta\sum_{i=1}^{n}x_i^{*}u_i+\sum_{i=1}^{n}\varepsilon_i u_i+\sum_{i=1}^{n}x_i^{*}v_i+\sum_{i=1}^{n}u_i v_i}{\sum_{i=1}^{n}\left(x_i^{*}+u_i\right)^2}$$

Taking probability limit

$$p\lim\hat{\beta}=\frac{\beta\sum_{i=1}^{n}x_i^{*2}/n+\sum_{i=1}^{n}x_i^{*}\varepsilon_i/n+\beta\sum_{i=1}^{n}x_i^{*}u_i/n+\sum_{i=1}^{n}\varepsilon_i u_i/n+\sum_{i=1}^{n}x_i^{*}v_i/n+\sum_{i=1}^{n}u_i v_i/n}{\sum_{i=1}^{n}\left(x_i^{*}+u_i\right)^2/n}=\frac{\beta\sigma_{x^*}^2}{\sigma_{x^*}^2+\sigma_u^2}$$

Grading Policy: Some credits are given for wrong answers but correct steps.

Question 6 (Wage Equation) [15 points] – Suppose the following wage equation is the true model:

$w_i=\beta S_i^{*}+\gamma A_i+\varepsilon_i$, $E\left(\varepsilon_i\mid S_i^{*},A_i\right)=0$, and $\operatorname{var}\left(\varepsilon_i\mid S_i^{*},A_i\right)=\sigma_\varepsilon^2$

In words, log wage w_i is a function of schooling S_i^{*} and ability A_i. All variables have zero mean. Unluckily, you do not observe A_i, and, worse still, schooling is measured with error $S_i=S_i^{*}+v_t$, $E\left(v_i\mid S_i^{*},A_i\right)=0$, $E\left(v_i^2\mid S_i^{*},A_i\right)=\sigma_v^2$, and $E\left(\varepsilon_i v_i\mid S_i^{*},A_i\right)=0$. So you are estimating $w_i=\beta S_i+u_i$. Prove that the probability limit of the OLS estimator $\hat{\beta}$ in this regression can be either larger than or smaller than β.

Answer

Solve for the OLS estimator:

$$\hat{\beta} = \frac{\sum_{i=1}^{n} S_i w_i}{\sum_{i=1}^{n} S_i^2} = \frac{\sum_{i=1}^{n}\left(S_i^* + v_i\right)\left(\beta S_i^* + \gamma A_i + \varepsilon_i\right)}{\sum_{i=1}^{n}\left(S_i^* + v_i\right)^2}$$

$$= \frac{\beta\sum_{i=1}^{n} S_i^{*2} + \gamma\sum_{i=1}^{n} S_i^* A_i + \sum_{i=1}^{n} S_i^*\varepsilon_i + \beta\sum_{i=1}^{n} S_i^* v_i + \gamma\sum_{i=1}^{n} v_i A_i + \sum_{i=1}^{n} v_i\varepsilon_i}{\sum_{i=1}^{n} S_i^{*2} + \sum_{i=1}^{n} v_i^2 + 2\sum_{i=1}^{n} S_i^* v_i}$$

Divide all by the sample size and take probability limit:

$$p\lim\hat{\beta}$$

$$= p\lim\frac{\beta\sum_{i=1}^{n} S_i^{*2}/n + \gamma\sum_{i=1}^{n} S_i^* A_i/n + \sum_{i=1}^{n} S_i^*\varepsilon_i/n + \beta\sum_{i=1}^{n} S_i^* v_i/n + \gamma\sum_{i=1}^{n} v_i A_i/n + \sum_{i=1}^{n} v_i\varepsilon_i/n}{\sum_{i=1}^{n} S_i^{*2}/n + \sum_{i=1}^{n} v_i^2/n + 2\sum_{i=1}^{n} S_i^* v_i/n}$$

$$= \frac{\beta Q_{S^*} + \gamma Q_{S^*A}}{Q_{S^*} + \sigma_v^2}$$

$$= \beta - \frac{\beta\sigma_v^2}{Q_{S^*} + \sigma_v^2} + \frac{\gamma Q_{S^*A}}{Q_{S^*} + \sigma_v^2}$$

We would expect $\gamma > 0$ and $Q_{S^*A} > 0$. We may asymptotically bias the return to schooling upward or downward, depending on the relative size of the omitted variable bias and measurement error bias.

Grading Policy: Some credits are given for wrong answers but correct steps.

Question 7 (Demand-Supply Model) [15 points] – Consider the simple demand-supply model:

$q_{d,t} = \alpha_1 p_t + \alpha_2 x_t + \varepsilon_{d,t}$ (Quantity demanded is a function of price and income)

$q_{s,t} = \beta_1 p_t + \varepsilon_{s,t}$ (Quantity supplied is a function of price)

$q_t = q_{d,t} = q_{s,t}$ (Market equilibrium)

The two shocks are serially uncorrelated, and have the properties:

$E\left(\varepsilon_{d,t} \mid x_t\right) = E\left(\varepsilon_{s,t} \mid x_t\right) = 0$, $E\left(\varepsilon_{d,t}{}^2 \mid x_t\right) = \sigma_d^2$, $E\left(\varepsilon_{s,t}{}^2 \mid x_t\right) = \sigma_s^2$, $E\left(\varepsilon_{d,t}\varepsilon_{s,t} \mid x_t\right) = 0$

a) Derive the IV estimator for β_1. (3 points)

b) Derive the 2SLS estimator for β_1. (3 points)

c) Show that the answers to a) and b) are equivalent. (3 points)

d) Solve for the reduced form of the system. (3 points)

e) Based on d), describe briefly how you would estimate β_1 by indirect least squares. (3 points)

Answer

a) Using income as an instrument, we have $\hat{\beta}^{IV} = \frac{\sum_{t=1}^{T} x_t q_t}{\sum_{t=1}^{T} x_t p_t}$

b) First we run p_t on x_t and get the fitted value $\hat{p}_t = \frac{x_t \sum_{t=1}^{T} x_t p_t}{\sum_{t=1}^{T} x_t^2}$, and second we run q_t on the fitted value to get $\hat{\beta}^{2SLS} = \frac{\sum_{t=1}^{T} \hat{p}_t q_t}{\sum_{t=1}^{T} \hat{p}_t^2}$.

c) Using matrix notation: $\hat{\mathbf{p}} = \mathbf{x}(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{p}$ and

$$\hat{\beta}^{2SLS} = (\hat{\mathbf{p}}'\hat{\mathbf{p}})^{-1}\hat{\mathbf{p}}'\mathbf{q} = \left(\mathbf{p}'\mathbf{x}(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{x}(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{p}\right)^{-1}\mathbf{p}'\mathbf{x}(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{q}$$

$$= \left(\mathbf{p}'\mathbf{x}(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{p}\right)^{-1}\mathbf{p}'\mathbf{x}(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{q}$$

$$= (\mathbf{x}'\mathbf{p})^{-1}\left(\mathbf{p}'\mathbf{x}(\mathbf{x}'\mathbf{x})^{-1}\right)^{-1}\mathbf{p}'\mathbf{x}(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{q}$$

$$= (\mathbf{x}'\mathbf{p})^{-1}\mathbf{x}'\mathbf{q} = \hat{\beta}^{IV}$$

d) Doing algebra:

$$p_t = \frac{\alpha_2 x_t + \varepsilon_{d,t} - \varepsilon_{s,t}}{\beta_1 - \alpha_1} = \pi_1 x_t + v_{1,t}$$

$$q_t = \frac{\beta_1 \alpha_2 x_t + \beta_1 \varepsilon_{d,t} - \alpha_1 \varepsilon_{s,t}}{\beta_1 - \alpha_1} = \pi_2 x_t + v_{2,t}$$

e) Estimate the two reduced-form equations in d) by OLS and obtain $\hat{\pi}_1$ and $\hat{\pi}_2$, the indirect least squares estimator for the supply slope is simply $\hat{\pi}_2 / \hat{\pi}_1 = \hat{\beta}_1$

Grading Policy: Some credits are given for wrong answers but correct steps.