ECON582 Econometrics III – Final Exam

31st May 2007, 1:30 pm-3:30 pm

It is a closed-book exam, and you do not need any statistical table

Answer ALL questions and keep your answer short

Total points=130

Question 1 (Warming Up) [5 points]: Explain briefly why for a stationary time series the autocovariance function is symmetric, i.e. $\lambda_k = \lambda_{-k}$.

Answer: With stationarity, $\operatorname{cov}(y_t, y_s)$ is a function of $|t-s|$, but not of t or s. So for any $k = t - s$, we have $\operatorname{cov}(y_t, y_s) = \operatorname{cov}(y_s, y_t)$.

Note: You have to mention how the result is related to the definition of stationarity.

Question 2 (Modeling Count Data with Maximum Likelihood) [20 points]: You are modeling a random variable y_i that only takes non-negative integer values (0, 1, 2,…).

a) Which of the following functions can serve as a probability density function (pdf) for y_i? You must explain your answer to get credits. [10 points]

$$f(y_i \mid \lambda) = \frac{e^{-\lambda} y_i}{y_i!}$$

$$f(y_i \mid \lambda) = \frac{-e^{\lambda} \lambda y_i}{y_i!}$$

$$f(y_i \mid \lambda) = \frac{e^{-\lambda} \lambda^{y_i}}{y_i!}$$

Answer: For a function to be a pdf, we require the function to be **positive** when evaluated at each possible value of y_i, and we require the function to **sum to one**. The first one cannot: $\sum_{y_i=0}^{\infty} f(y_i \mid \lambda) = e^{-\lambda}\left(1+1+\frac{3}{6}+\frac{4}{24}+\cdots\right) \neq 1$, and the second one is negative evaluated at each possible value of y_i. Only the third one qualifies (it is the Poisson distribution, by the way): $\sum_{y_i=0}^{\infty} f\left(y_i \mid \lambda\right) = e^{-\lambda} \sum_{y_i=0}^{\infty} \frac{\lambda^{y_i}}{y_i!} = e^{-\lambda} e^{\lambda} = 1$.

Note: You have to mention the two features of a pdf: 1) it can't be negative and 2) it sums to one, and you also to mention why the other two can't be pdf. If you only pick the correct pdf, you get 5 points.

b) For the probability density function(s) you choose in a), derive the maximum likelihood estimator for λ and the information matrix (since we only have one parameter, the information matrix is just a scalar), given a sample of size n. No credits are given for this part if you have chosen the wrong function(s). [10 points]

Answer: First get the joint density of the sample and take log:

$$\ln L = \sum_{i=1}^{n} \ln f\left(y_i \mid \lambda\right) = -n\lambda + \ln \lambda \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \ln\left(y_i!\right)$$

Then differentiate and solve for the ML estimator:

$$\frac{d \ln L}{d\lambda} = -n + \frac{\sum_{i=1}^{n} y_i}{\lambda} = 0 \Rightarrow \hat{\lambda} = \frac{\sum_{i=1}^{n} y_i}{n} = \overline{y}$$

Then differentiate again:

$$\frac{d^2 \ln L}{d\lambda d\lambda} = -\frac{\sum_{i=1}^{n} y_i}{\lambda^2}$$

Take expectation conditional of the true parameter value, and make use of the fact that $E\left(y_i\right) = \lambda$, we have $I\left(\lambda\right) = \frac{n}{\lambda}$.

Note: I accept some small calculation mistakes here.

Question 3 (Time Series Models with Maximum Likelihood) [20 points]

a) Consider the AR(1) process:

$$y_t = c + \phi y_{t-1} + \varepsilon_t,\ \varepsilon_t \sim \text{i.i.d. } N\left(0, \sigma^2\right)$$

Given a sample $(y_1, y_2, ..., y_T)$ of length T from the above process, explain in detail how you would derive the log likelihood function for the whole sample. [10 points]

Answer: The density of the first observation:

$$f\left(y_1; c, \phi, \sigma^2\right) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2/\left(1-\phi^2\right)}} \exp\left[\frac{-\left(y_1 - \left(c/\left(1-\phi\right)\right)\right)^2}{2\sigma^2/\left(1-\phi^2\right)}\right]$$

Conditional on the first observation, we have the density for the second observation:

$$f\left(y_2 \mid y_1; c, \phi, \sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-\left(y_2 - c - \phi y_1\right)^2}{2\sigma^2}\right]$$

Likewise for the rest of the sample. The log-likelihood function is simply:

$$\ln L\left(c, \phi, \sigma^2\right) = \ln f\left(y_1; c, \phi, \sigma^2\right) + \sum_{t=2}^{T} \ln f\left(y_t \mid y_{t-1}; c, \phi, \sigma^2\right)$$

$$= -\frac{T}{2}\ln 2\pi - \frac{1}{2}\ln\left(\sigma^2/\left(1-\phi^2\right)\right) - \frac{\left(y_1 - c/\left(1-\phi\right)\right)^2}{2\sigma^2/\left(1-\phi^2\right)} - \frac{T-1}{2}\ln\sigma^2 - \sum_{t=2}^{T}\frac{\left(y_t - c - \phi y_{t-1}\right)^2}{2\sigma^2}$$

Note: If you only describe correctly how to get the function without writing it down, you will get 5 points. Since I mention it in class, no credit for correct steps but wrong answer!

b) Consider the MA(1) process:

$$y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}, \quad \varepsilon_t \sim \text{i.i.d. } N\left(0, \sigma^2\right)$$

Given a sample $(y_1, y_2, ..., y_T)$ of length T from the above process and that $\varepsilon_0 = 0$, explain in detail how you would derive the log likelihood function for the whole sample. [10 points]

Answer: Given $\varepsilon_0 = 0$, we can write down the density of the first observation:

$$f\left(y_1 \mid \varepsilon_0 = 0; \mu, \theta, \sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-\left(y_1 - \mu\right)^2}{2\sigma^2}\right]$$

Given the first observation and $\varepsilon_0 = 0$, we know $\varepsilon_1 = y_1 - \mu$, and so we can write down the density for the second observation:

$$f\left(y_2 \mid y_1, \varepsilon_0 = 0; \mu, \theta, \sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-\left(y_2 - \mu - \theta\varepsilon_1\right)^2}{2\sigma^2}\right]$$

Keep doing this and we can write down the log likelihood function:

$$\ln L\left(\mu,\phi,\sigma^2\right)=\ln f\left(y_1 \mid \varepsilon_0=0;\mu,\phi,\sigma^2\right)+\sum_{t=2}^{T}\ln f\left(y_t \mid y_{t-1};\mu,\phi,\sigma^2\right)$$

$$=-\frac{T}{2}\ln 2\pi-\frac{1}{2}\ln\left(\sigma^2\right)-\sum_{t=1}^{T}\frac{\varepsilon_t^{\,2}}{2\sigma^2}$$

Note: If you only describe correctly how to get the function without writing it down, you will get 5 points. Some effort of getting the function is also worth 5 points.

Question 4 (Simultaneous Equations Model) [20 points]

Consider the following system:

$$y_1=\gamma_1 y_2+\beta_1 x+\varepsilon_1$$

$$y_2=\gamma_2 y_1+\beta_2 x+\varepsilon_2$$

Conditional on the exogenous variable x, the structural disturbances have the distribution: $\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}\sim i.i.d.N\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}\right)$.

a) Is the system identified (or partially identified) if you have the restrictions $\gamma_1=0$ and $\sigma_{12}=0$? If so, describe how you would estimate the equation(s) in EViews if you are given data on y_1, y_2, and x. [10 points]

Answer: Both equations are identified. The first equation just becomes a classical linear regression, and can be estimated by OLS. The second equation also becomes a classical linear regression too, as now y_1 is uncorrelated to ε_2, as the two shocks are uncorrelated, and so the equation can be estimated by OLS.

Note: If you use IV (which is not needed and inefficient) to estimate the second equation: 7 points. If you only get the first equation right, 5 points.

b) Is the system identified (or partially identified) if you have the restriction $\beta_1=0$? If so, describe how you would estimate the equation(s) in EViews if you are given data on y_1, y_2, and x. [10 points]

Answer: Only the first equation is identified. Using x as instrument for y_2 in the first equation, we obtain the IV estimate for γ_1.

Note: It is an all-or-nothing question.

Question 5 (Dynamic Regression Model) [20 points]

Consider the following model:

$$y_t = 2 + 0.9x_t + 0.8x_{t-1} + 0.7x_{t-2} + 0.6x_{t-3} + 0.5x_{t-4} + 0.4x_{t-5} + 0.3x_{t-6} + 0.2x_{t-7} + 0.1x_{t-8} + \varepsilon_t,$$

where $E\left(\varepsilon_t \mid x_t, x_{t-1}, \ldots\right) = 0$ and $E\left(\varepsilon_t^{\,2} \mid x_t, x_{t-1}, \ldots\right) = \sigma^2$

a) If for many, many periods the exogenous variable has been staying at the equilibrium value $\bar{x} = 2$, calculate the equilibrium value for y. [5 points]

Answer: The equilibrium multiplier is 0.9+0.8+…+0.1 = 4.5. So the equilibrium value for y is 2+4.5*2 = 11.

b) Calculate the equilibrium multiplier, and calculate the lag weights for the RHS variables $\left(x_t, \ldots, x_{t-8}\right)$. [5 points]

Answer: It is 4.5 as calculated above. The lag weights are 0.9/4.5, 0.8/4.5,…. = 0.2, 0.18, 0.16, 0.13, 0.11, 0.09, 0.07, 0.04, 0.02 approximately.

c) Calculate the median lag and the mean lag. [5+5 points]

Answer: The mean lag is 2.67 and the median lag is 2.

Note: This question is purely arithmetical, and it is all-or-nothing.

Question 6 (AR(2) Process) [20 points]

Consider the following model:

$y_t = 3 + 0.9y_{t-1} - 0.2y_{t-2} + \varepsilon_t$, where ε_t is white noise with variance σ^2

a) Is the process stationary? Explain. [5 points]

Answer: Using the three conditions: 0.9-1>-0.2, 1+0.9>-0.2 and -0.2>-1. The process is stationary.

Note: Either you find the roots, or state the three conditions.

b) Calculate $E\left(y_t\right), \gamma_0, \gamma_1, \gamma_2, \gamma_3$. [10 points]

Answer: We know $E\left(y_t\right) = \dfrac{c}{1-\phi_1-\phi_2} = 10$ and $\gamma_0 = \dfrac{\left(1-\phi_2\right)\sigma^2}{\left(1+\phi_2\right)\left[\left(1-\phi_2\right)^2 - \phi_1^{\,2}\right]} = 2.381\sigma^2$

From the Yule-Walker Equations:

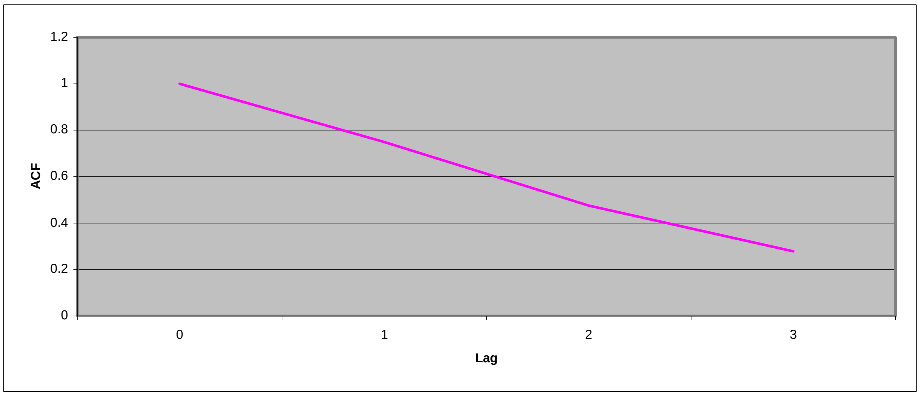
$$\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1 \Rightarrow \gamma_1 = \frac{\phi_1\gamma_0}{1-\phi_2} = 1.786\sigma^2$$

$$\gamma_2 = \phi_1\gamma_1 + \phi_2\gamma_2 = 1.131\sigma^2$$

$$\gamma_3 = \phi_1\gamma_2 + \phi_2\gamma_1 = 0.661\sigma^2$$

Note: Wrong numbers but correct steps get 8 points.

c) Plot the autocorrelation function for lags 1, 2, and 3. [5 points]



Note: Wrong due to wrong calculation above, get 2 points.

Question 7 (Random Walk Plus Noise Model) [25 points]

Consider a time series $\{y_t\}$ that consists of a random walk plus an uncorrelated white noise:

$$\alpha_t = \alpha_{t-1} + \varepsilon_{1,t}$$

$$y_t = \alpha_t + \varepsilon_{2,t}$$

$$\begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right)$$

a) Define the signal-to-noise ratio and show that the autocorrelations for $(1-L)\,y_t$ are functions of that ratio. [5 points]

Answer: Signal-to-noise ratio $S = \frac{\sigma_1^2}{\sigma_2^2}$. Rewriting the model:

$$(1-L)\,y_t = \varepsilon_{1,t} + \varepsilon_{2,t} - \varepsilon_{2,t-1}$$

So we have $\gamma_0 = \sigma_1^2 + 2\sigma_2^2$ and $\gamma_1 = -\sigma_2^2$, and the first-order autocorrelation is simply

$$\rho_1 = \frac{-\sigma_2^2}{\sigma_1^2 + 2\sigma_2^2} = \frac{-1}{S+2}.$$

Note: The ratio itself is worth 2 points.

b) Derive the reduced form IMA(1,1) model for this time series. Show clearly how the parameters in the reduced form and the original models are related. For the two solutions for the IMA(1,1) model, show that one is invertible and one is not. [10 points]

Answer: Consider an IMA(1,1) model:

$$(1-L)y_t = u_t - \theta u_{t-1}$$

The moments are $\gamma_0 = \sigma_u^2(1+\theta^2)$ and $\gamma_1 = -\theta\sigma_u^2$. Matching the moments:

$\sigma_1^2 + \sigma_2^2 = \sigma_u^2(1+\theta^2)$ and $\sigma_2^2 = \theta\sigma_u^2$ we can solve out the two equations for the two unknowns θ and σ_u^2.

Solving out we have $\theta = \dfrac{S+2 \pm \sqrt{(S+2)^2 - 4}}{2} = 1 + \dfrac{S \pm S\sqrt{1+4/S}}{2}$, and only the solution

$\theta = 1 + \dfrac{S - S\sqrt{1+4/S}}{2} < 1$ is invertible.

Note: Some scribble of math is worth 2 points.

c) Derive the one-step-ahead forecast of y_t based on past observations $\{y_{t-1}, y_{t-2}, \ldots\}$. [5 points]

Answer: Rewrite the IMA(1,1) model

$$(1-L)y_t = (1-\theta L)u_t$$

$$(1-\theta L)^{-1}(1 - L + \theta L - \theta L)y_t = u_t$$

$$y_t = \frac{1-\theta}{(1-\theta L)} y_{t-1} + u_t$$

Taking expectation:

$$E(y_t \mid y_{t-1}, y_{t-2}, \ldots) = \hat{y}_{t|t-1} = \frac{1-\theta}{(1-\theta L)} y_{t-1}$$

It can be further rewritten as

$$\hat{y}_{t|t-1} = (1-\theta)y_{t-1} + \theta \hat{y}_{t-1|t-2}$$

Note: It is an all-or-nothing question.

d) Express the forecast revision as a function of the "gain". [5 points]

Answer: Simply rewrite the above solution:

$$\hat{y}_{t|t-1} - \hat{y}_{t-1|t-2} = (1-\theta)\left(y_{t-1} - \hat{y}_{t-1|t-2}\right) = G\left(y_{t-1} - \hat{y}_{t-1|t-2}\right)$$

Note: It is an all-or-nothing question.